UBC Math 403(101) **Midterm Test** 19 October 2012

Write your answers in the booklet provided. Start each solution on a separate page.
Any number of notes in your own handwriting are <u>allowed</u>*; anything else is* <u>forbidden</u>.

SHOW ALL YOUR WORK!!

[15] **1.** Consider the controlled nonlinear system below. The state $\mathbf{y}$ evolves in $\mathbb{R}^3$; the control, u, is scalar-valued; α, β, γ are constants.

$$\dot{y}_1 = -y_2 y_3 + \alpha u, \qquad \dot{y}_2 = y_1 y_3 + \beta u, \qquad \dot{y}_3 = -y_1 y_2 + \gamma u.$$

Observe that for any fixed $k > 0$, the constant vector $\overline{\mathbf{y}} = (0, k, 0)$ provides a solution for this system corresponding to the constant input $\overline{u} \equiv 0$.

(a) Linearize the system about the reference solution $(\overline{u}, \overline{\mathbf{y}})$, to obtain a new system in which $\mathbf{x}$ approximates the true system's deviation from $\overline{\mathbf{y}}$ via

$$(**) \qquad \dot{\mathbf{x}} = A\mathbf{x} + Bu.$$

(b) Analyze the stability of the uncontrolled system $\dot{\mathbf{x}} = A\mathbf{x}$. (Take A from $(**)$.)

(c) Characterize the choices of α, β, γ for which system $(**)$ is controllable.

[25] **2.** Consider this single-input system with a three-dimensional state $\mathbf{x} = [\, x_1 \quad x_2 \quad x_3 \,]^T$:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \qquad \text{a.e. } t. \qquad (*)$$

(a) Is this system controllable? Justify your answer.

(b) Give a precise geometric description of the attainable sets $\mathcal{A}(t; 0)$ for $t > 0$.

(c) A customer asks your firm to supply a "feedback matrix" $F = [\, a \quad b \quad c \,]$ such that substituting $u = F\mathbf{x}$ in $(*)$ will produce an autonomous system for which every trajectory $\mathbf{x}$ obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Choose one of the following responses, and support it with detailed calculations or a clear proof, as appropriate:

(i) "Here is a matrix of the type you requested. It even has integer entries! Please watch your mailbox for our invoice."

(ii) "A matrix meeting your specifications does not exist. We cannot accept your order."

[10] **3.** (a) Prove: *For any closed convex set $S \subseteq \mathbb{R}^n$ and any point s_0 in S, the set $K = N_S(s_0)$ (i.e., the collection of outward normals to S at s_0) is also convex.* Include clear definitions of "convex set" and "outward normal".

(b) Sketch the set $S \subseteq \mathbb{R}^2$ defined below, and give a precise characterization of the set $N_S(s_0)$ at each point s_0 on the boundary of S:

$$S = \left\{ (x, y) \,:\, x \geq 0,\ x^3 - 1 \leq y \leq 1 - x^2 \right\}.$$

(Suggestion: Write (x, y) for typical points in S, and (p, q) for typical elements of N_S.)