[15] **1.** (a) For the deviation vector $\mathbf{x} = \mathbf{y} - \overline{\mathbf{y}}$, we have $(y_1, y_2, y_3) = (x_1, x_2 + k, x_3)$, so the system becomes

$$\begin{aligned} \dot{x}_1 &= -(x_2+k)x_3 + \alpha u = -kx_3 - x_2x_3 + \alpha u \\ \dot{x}_2 &= x_1x_3 + \beta u = x_1x_3 + \beta u \\ \dot{x}_3 &= -x_1(x_2+k) + \gamma u = -kx_1 - x_1x_2 + \gamma u. \end{aligned}$$

Linearization around $\mathbf{x} = (0,0,0)$ is equivalent to discarding all terms of order exceeding 1. The linearized system (re-using variable names) is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} u.$$

(b) For the system matrix above, namely

$$A = \begin{bmatrix} 0 & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & 0 \end{bmatrix},$$

the eigenvalues are roots of the characteristic polynomial, which is

$$p(\lambda) = \det(\lambda I - A) = \lambda \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} + k \begin{vmatrix} 0 & \lambda \\ k & 0 \end{vmatrix} = \lambda^3 - k^2\lambda = \lambda(\lambda - k)(\lambda + k).$$

Since $k > 0$, the eigenvalue $\lambda = k > 0$ shows that the uncontrolled system is *unstable*.

(c) The controllability matrix is

$$C = \left[B \,\middle|\, AB \,\middle|\, A(AB) \right] = \begin{bmatrix} \alpha & -k\gamma & k^2\alpha \\ \beta & 0 & 0 \\ \gamma & -k\alpha & k^2\gamma \end{bmatrix}.$$

Expanding along the middle row gives

$$\det(C) = -\beta \begin{vmatrix} -k\gamma & k^2\alpha \\ -k\alpha & k^2\gamma \end{vmatrix} = -\beta\left[-k^3\gamma^2 + k^3\alpha^2 \right] = k^3\beta(\gamma^2 - \alpha^2).$$

Controllability is equivalent to $\det(C) \neq 0$. That is,

$$\left[\text{the pair } (A,B) \text{ is controllable}\right] \iff \left[\beta \neq 0 \text{ and } \alpha^2 \neq \gamma^2\right].$$

[25] **2.** (a) We test for controllability by filling in the matrix

$$\mathcal{C} = \left[B \,|\, AB \,|\, A^2B\right] = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

This clearly has rank less than 3, so the system is **not controllable.**

(b) In the absence of control constraints, we proved in class that

$$\begin{aligned} \mathcal{A}(t;0) = \mathrm{Im}(\mathcal{C}) &= \left\{\mathcal{C}w \,:\, w \in \mathbb{R}^3\right\} \\ &= \left\{(2w_2 + 4w_3,\, 0,\, w_1 + w_2 + 5w_3) \,:\, w \in \mathbb{R}^3\right\} \\ &= \left\{(x_1, 0, x_3) \,:\, x_1, x_3 \in \mathbb{R}\right\}. \end{aligned}$$

This is the plane in (x_1, x_2, x_3)-space defined by the equation $x_2 = 0$.

(c) Taking $u = Fx$ with $F = [a\ b\ c]$ produces the autonomous system $\dot{x} = (A + BF)x$, in which the coefficient matrix is

$$A + BF = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ a+2 & b & c+1 \end{bmatrix}.$$

This has characteristic polynomial

$$\begin{aligned} p(\lambda) = \det(\lambda I - (A + BF)) &= (\lambda - 1)\begin{vmatrix} \lambda+1 & 0 \\ -b & \lambda - c - 1 \end{vmatrix} + (-2)\begin{vmatrix} 0 & \lambda+1 \\ -a-2 & -b \end{vmatrix} \\ &= (\lambda-1)(\lambda+1)(\lambda-c-1) - 2(\lambda+1)(a+2) = (\lambda+1)(\lambda^2 - (c+2)\lambda - 2a + c - 3). \end{aligned}$$

One eigenvalue, $\lambda = -1$, is independent of the choice of F. The other two, by the quadratic formula, satisfy

$$2\lambda = (c+2) \pm \sqrt{(c+2)^2 + 8a - 4c + 12}.$$

To make sure both of these satisfy $\Re e(\lambda) < 0$, the first step is to make sure $c + 2 < 0$. So try $c = -3$. This gives

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{25 + 8a}.$$

Any integer $a \leq -4$ will lead to $\Re e(\lambda) = -3/2$ for both cases. The choice $a = -3$ is no good, however, because it gives a zero eigenvalue; larger choices of a are even worse.

The value of b makes no difference to the eigenvalues of $A + BF$, so we might as well choose $b = 0$. Then we can respond with option (i): the matrix $F = [-4 \quad 0 \quad -3]$ leads to a feedback setup where $A + BF$ has eigenvalues $-1, (-1 \pm i\sqrt{7})/2$, and therefore every trajectory has limit $\mathbf{0}$ as $t \to \infty$.

Of course, there are infinitely many possibilities for the matrix F. A triple (a, b, c) will produce the desired system behaviour if and only if $3 + 2a < c < -2$.

[10] **3.** (a) In the given setup, suppose w_0 and w_1 are arbitrary elements of K. Fix any $\alpha \in (0,1)$ and consider $w_\alpha = (1-\alpha)w_0 + \alpha w_1$: it suffices to prove that $w_\alpha \in K$.

The definition of $K = N_S(s_0)$ entails

$$\begin{array}{lll} (0) & \forall s \in S, & \langle w_0\,,\, s - s_0 \rangle \leq 0, \\ (1) & \forall s \in S, & \langle w_1\,,\, s - s_0 \rangle \leq 0. \end{array}$$

Multiply (0) by $(1-\alpha) > 0$ and (1) by $\alpha > 0$ (this preserves both inequalities) and add the results:

$$\forall s \in S, \qquad 0 \geq \langle (1-\alpha)w_0 + \alpha w_1\,,\, s - s_0 \rangle = \langle w_\alpha\,,\, s - s_0 \rangle\,.$$

This shows that $w_\alpha \in N_S(s_0) = K$, as required.

(b) The given set has a flat left side, a curved parabolic upper boundary, and a curved cubic lower boundary. A good sketch, like the one below, makes it possible to handle all boundary points away from the corners essentially by inspection.

On the flat left side, where the generic point is $(0,y)$ with $|y| < 1$,

$$N_S(0,y) = \{t(-1,0) \,:\, t \geq 0\} \quad \text{for} \quad -1 < y < 1.$$

On the curved top in the first quadrant, where $y = 1 - x^2$ and $0 < x < 1$, knowing that $y' = -2x$ gives a vector $(1, -2x)$ tangent to the curve. So an outward normal vector is $(2x, 1)$, and consequently

$$N_S(x, 1-x^2) = \{t(2x,1) \,:\, t \geq 0\} \quad \text{for} \quad 0 < x < 1.$$

On the curved bottom in the fourth quadrant, where $y = x^3 - 1$ and $0 < x < 1$, knowing that $y' = 3x^2$ gives a vector $(1, 3x^2)$ tangent to the curve. Hence an outward normal is $(3x^2, -1)$, and consequently

$$N_S(x, x^3-1) = \left\{t(3x^2,-1) \,:\, t \geq 0\right\} \quad \text{for} \quad 0 < x < 1.$$

At the vertices, two smooth sides collide and each prescription provides one normal vector. Each curved boundary has a vertical outward normal at the point where it hits $x = 0$, and we get a horizontal outward normal from the flat side of S. The result of part (a) implies that the cone of vectors we seek must be convex. It's clear that simply filling in all the vectors between the "obvious" two found above produces *all* the desired normals. Consequently

$$\begin{aligned} N_S(0,-1) &= \{(p,q) \,:\, p \leq 0 \text{ and } q \leq 0\}\,, \\ N_S(0,1) &= \{(p,q) \,:\, p \leq 0 \text{ and } q \geq 0\}\,, \\ N_S(1,0) &= \left\{t(1,r) \,:\, t \geq 0 \text{ and } -\frac{1}{3} \leq r \leq \frac{1}{2}\right\}. \end{aligned}$$

The question asks only for a sketch of the set S, but here is a little more—namely, a colour-coded illustration of the normal cone mapping associated with S.