A Fixed-Time Problem with Rocket-Car Dynamics

UBC Math 403 Lecture Notes by Philip D. Loewen

Problem Statement. A point (x_f, y_f) in state space is given, and used to define the objective function

$$\ell(x_1, x_2) = \tfrac{1}{2}(x_1 - x_f)^2 + \tfrac{1}{2}(x_2 - y_f)^2.$$

We start the rocket car at the origin of state space and minimize the ℓ-value of its state at time $T = 2$. A concise statement of this problem appears below.

$$\begin{array}{lll} \text{minimize} & \tfrac{1}{2}(x_1(T) - x_f)^2 + \tfrac{1}{2}(x_2(T) - y_f)^2 & \\ \text{subject to} & \dot{x}_1 = x_2 \qquad \text{a.e. } t \in [0,T], & x_1(0) = 0, \\ & \dot{x}_2 = u \qquad \text{a.e. } t \in [0,T], & x_2(0) = 0, \\ & u \in [-1,1], \qquad \text{a.e. } t \in [0,T]. & \end{array}$$

PreHamiltonian. $H(x_1, x_2, p_1, p_2, u) = p_1 x_2 + p_2 u.$

Transversality.

$$-p_1(T) = \left.\frac{\partial \ell}{\partial x_1}\right|_{\mathbf{x}(T)} = x_1(T) - x_f,$$
$$-p_2(T) = \left.\frac{\partial \ell}{\partial x_2}\right|_{\mathbf{x}(T)} = x_2(T) - y_f$$

Maximum Condition.

$$\widehat{u}(t) \in \arg\max_{v \in [-1,1]} H(\mathbf{x}(t), \mathbf{p}(t), v) = \operatorname{Sgn}(p_2(t)).$$

Costate Evolution. There must be constants m and b for which

$$-\dot{p}_1(t) = \frac{\partial H}{\partial x_1} = 0 \qquad \Longrightarrow \qquad p_1(t) = -m,$$
$$-\dot{p}_2(t) = \frac{\partial H}{\partial x_2} = p_1 \qquad \Longrightarrow \qquad p_2(t) = mt + b.$$

Degenerate Cases. Could $m = 0$ and $b = 0$? Yes, but only in special situations. These choices make $\mathbf{p}(T) = 0$, and in this case the transversality conditions require $(x_1(T), x_2(T)) = (x_f, y_f)$. That is, if the target point (x_f, y_f) can be hit exactly by the system state, then of course that will give the minimum. It's a bit uninteresting as a challenge, though. The problems we care most about are ones where the target (x_f, y_f) lies outside the attainable set at time T.

Could $m = 0$ with $b \neq 0$? Yes, but this is pretty special too. These choices make $\widehat{u}(t) = \operatorname{sgn}(b) =: \sigma$ a constant. Then the state obeys

$$x_1(t) = \frac{\sigma}{2} t^2, \quad x_2(t) = \sigma t,$$

and terminates at $\mathbf{x}(T) = \sigma(\frac{1}{2}T^2, T)$. This endpoint is compatible with the transversality conditions above only when

$$\begin{aligned} 0 = -p_1(T) &= x_1(T) - x_f = \frac{\sigma}{2}T^2 - x_f, \\ -b = -p_2(T) &= x_2(T) - y_f = \sigma T - y_f. \end{aligned}$$

Different choices of $b \neq 0$ give different possibilities here, but they are all rather special. The system defining extremality has a solution with $m = 0$ and $b > 0$ only if the target point lies on this vertical ray parametrized by b:

$$(x_1, x_2) = (\tfrac{1}{2}T^2, T) + (0, b), \qquad b > 0.$$

The symmetric ray captures all targets where one of these extremals can work with $b = -\beta < 0$:

$$(x_1, x_2) = -(\tfrac{1}{2}T^2, T) - (0, \beta), \qquad \beta > 0.$$

Nondegenerate Cases. For a target point (x_f, y_f) that lies outside the attainable set $\mathcal{A} = \mathcal{A}(T; \mathbf{0}, [-1, 1])$ and not on one of the special rays identified above, extremality requires $m \neq 0$. Thus we can write

$$p_2(t) = mt + b = m(t + b/m) = m(t - \theta), \qquad \text{where} \qquad \theta \stackrel{\text{def}}{=} -b/m.$$

Let $\sigma = -\operatorname{sgn}(m)$ so that (at all instants except $t = \theta$)

$$\widehat{u}(t) = \operatorname{sgn}(p_2(t)) = \operatorname{sgn}(-\sigma(t - \theta)) = \begin{cases} \sigma, & \text{for } 0 < t < \theta, \\ -\sigma, & \text{for } \theta < t < T. \end{cases}$$

Now integrating the dynamics, using the initial condition and insisting on continuity at $t = \theta$, gives

$$\begin{array}{lll} \text{for } 0 \le t \le \theta: & x_2(t) = \sigma t, & x_1(t) = \dfrac{\sigma}{2}t^2, \\ \text{when } t = \theta: & x_2(\theta) = \sigma\theta, & x_1(\theta) = \dfrac{\sigma}{2}\theta^2, \\ \text{for } \theta \le t \le T: & x_2(t) = \sigma(2\theta - t), & x_1(t) = \sigma\theta^2 - \dfrac{\sigma}{2}(2\theta - t)^2, \\ \text{when } t = T: & x_2(T) = \sigma\left[2\theta - T\right], & x_1(T) = \sigma\left[2T\theta - \tfrac{1}{2}T^2 - \theta^2\right]. \end{array}$$

We must reconcile these outcomes with the transversality conditions:

$$\begin{aligned} m = &-p_1(T) = x_1(T) - x_f = \sigma\left[2T\theta - \tfrac{1}{2}T^2 - \theta^2\right] - x_f, \\ -m(T - \theta) = &-p_2(T) = x_2(T) - y_f = \qquad \sigma\left[2\theta - T\right] - y_f, \end{aligned}$$

Using the first equation to eliminate m from the second leads to a cubic equation for the switching time θ:

$$\sigma\left(3T\theta^2 - \frac{5}{2}T^2\theta - \theta^3 + \frac{1}{2}T^3 + T - 2\theta\right) + (T - \theta)x_f + y_f = 0.$$

A Specific Case. When $T = 2$ and $(x_f, y_f) = (3, 1)$, the equation above becomes

$$\sigma\left(6\theta^2 - 10\theta - \theta^3 + 6 - 2\theta\right) + 7 - 3\theta = 0.$$

When $\sigma = -1$ it has no solution in the interval $[0, 2]$, but when $\sigma = +1$ the unique solution is $\widehat{\theta} = 1.67781$. This gives the optimal final point $\mathbf{x}(2) = (1.89620, 1.35563)$, the costate parameters $m = -1.10380$ and $b = 1.85198$, and the final costate vector $\mathbf{p}(2) = (1.10380, -0.35563)$. The sketch shows the optimal state trajectory and final point, the target with some level sets for $\ell(x_1, x_2) = \frac{1}{2}(x_1 - x_f)^2 + \frac{1}{2}(x_2 - y_f)^2$, and the vector $\alpha\mathbf{p}(2)$ for $\alpha = 0.60$ (to make everything look nice).