[24] 1. **Short Answer Questions**. Each question is worth 3 points. Put a box around your final answer, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Evaluate $\int e^{2x} \sin 2x \, dx$. $= I$

Let $u = \sin(2x) \Rightarrow du = 2\cos(2x)\,dx$

$dv = e^{2x}dx \Rightarrow v = \frac{e^{2x}}{2}$.

$$I = \frac{e^{2x}}{2}\sin(2x) - \int \frac{e^{2x}}{\cancel{2}} \cdot \cancel{2}\cos(2x)\,dx$$

Let $U = \cos(2x) \Rightarrow dU = -2\sin(2x)dx$

$dV = e^{2x}dx \Rightarrow V = \frac{e^{2x}}{2}$

$$I = \frac{e^{2x}}{2}\sin(2x) - \frac{e^{2x}}{2}\cos(2x) - \underbrace{\int e^{2x}\sin(2x)dx}_{I}$$

$$\Rightarrow \boxed{I = \frac{1}{4}e^{2x}(\sin(2x) - \cos(2x)) + C.}$$

(b) Evaluate $\int \frac{x+1}{\sqrt{1-x^2}}\,dx$.

$$= \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{1}{\sqrt{1-x^2}}dx$$

$=$

Let $u = 1-x^2$

$du = -2x\,dx$

$$= \int \frac{-1/2\,du}{\sqrt{u}} + \arcsin(x)$$

$$= -\frac{1}{\cancel{2}}\cancel{2}u^{1/2} + \arcsin(x) + C$$

$$= \boxed{-\sqrt{1-x^2} + \arcsin(x) + C.}$$

(c) Find the average value of $f(x) = \sec^2(x)\, dx$ over the interval $[0, \pi/4]$.

$$f_{ave} = \frac{1}{(\pi/4 - 0)} \int_0^{\pi/4} \sec^2 x \, dx = \frac{1}{\pi/4} \tan x \Big|_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(\tan(\pi/4) - \tan(0)\right) = \boxed{\frac{4}{\pi}}$$ [redacted]

(d) Find $f'(x)$ where $f(x) = \int_{\ln(x)}^{e^x} \cos(x^2)\, dx$, where $x > 0$.

$$\boxed{f'(x) = \cos(e^{2x}) \cdot e^{x} - \cos(\ln^2(x)) \cdot \frac{1}{x}}$$

(Apply the Fundamental Theorem of Calculus and the chain rule)

(e) Evaluate $\int \frac{dx}{x^2 - 5x + 7}$ $= \int \frac{dx}{(x-5/2)^2 + 3/4} = \frac{4}{3}\int \frac{dx}{\frac{4}{3}(x-5/2)^2+1}$

$$= \boxed{\frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}(x-5/2)\right) + C}$$

(Use $\tan\theta = \frac{2}{\sqrt{3}}(x-5/2)$ if you don't see the arctangent immediately.)

(f) A variable force of $F(x) = kx$ newtons moves an object along a straight line when the object is x metres from the origin. If the work done in moving the object from 1 metre to 5 metres from the origin is 25 newton-metres, find the value of k.

$$25 = W = \int_1^5 F(x)\,dx = \int_1^5 kx\,dx = \frac{k}{2}x^2\Big|_1^5$$

$$= \frac{25}{2}k - \frac{1}{2}k = 12k.$$

$$\Rightarrow \quad 12k = 25$$

$$\Rightarrow \boxed{k = \frac{25}{12} \text{ N m}^{-1}}.$$

(g) Evaluate $\int_1^\infty xe^{-x}\,dx.$ $= \lim_{R\to\infty} \int_1^R xe^{-x}\,dx$

Let $u = x \Rightarrow du = dx$; $dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$$= \lim_{R\to\infty} \left[-x e^{-x} \Big|_1^R - \int_1^R -e^{-x}\,dx \right]$$

$$= \lim_{R\to\infty} \left(-Re^{-R} + \frac{1}{e} + \left[-e^{-x} \right]_1^R \right)$$

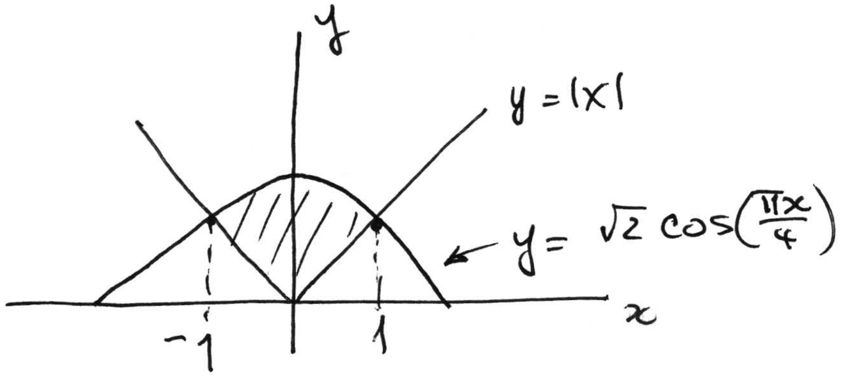
$$= \lim_{R\to\infty} \left(\cancel{-Re^{-R}}^{\to 0} + \frac{1}{e} - \cancel{e^{-R}}^{\to 0} + \frac{1}{e} \right) = \boxed{\frac{2}{e}}$$

(h) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^2 \cdot \frac{1}{n}.$

An integral that has this as a Riemann sum is $\int_1^2 x^2\,dx$ since $\Delta x = \frac{2-1}{n} = \frac{1}{n}$ and choosing right hand endpts on a partition of $[1,2]$ in n equal pieces gives $x_i^* = 1 + \frac{i}{n}$, $i = 1, 2, \ldots, n$.

Thus, the sum is $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$

[6] **2**. Find the area of the finite region R bounded by the curves $y = \sqrt{2}\cos(\pi x/4)$ and $y = |x|$.



Intersection points:

$\sqrt{2}\cos\left(\frac{\pi}{4}x\right) = |x|$

$\Rightarrow x = \pm 1$ since $\cos\left(\pm \pi/4\right) = \frac{1}{\sqrt{2}}$.

$$\text{Area} = \int_{-1}^{1} \left(\sqrt{2}\cos\left(\tfrac{\pi}{4}x\right) - |x|\right)dx = 2\int_0^1 \sqrt{2}\cos\left(\tfrac{\pi}{4}x\right) - x\,dx$$

by symmetry about the y-axis.

$$= \left[2\cdot\sqrt{2}\cdot\frac{4}{\pi}\sin\left(\frac{\pi}{4}x\right) - x^2\right]_0^1$$

$$= \frac{8\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{2}} - 1$$

$$= \boxed{\frac{8}{\pi} - 1}.$$

[6] **3**. Find the function f that satisfies $f(x) = 4x - \int_0^x f(t)\,dt$.

By the F.T.C., $f'(x) = 4 - f(x)$. (*)

We solve this DE by substitution:

Let $u = 4 - f(x)$

$\Rightarrow \frac{du}{dx} = -f'(x)$,

which transforms (*) into $-\frac{du}{dx} = u$

$\Rightarrow \frac{du}{dx} = -u$

$\Rightarrow u(x) = Ce^{-x}$ for some constant C.

$\Rightarrow 4 - f(x) = Ce^{-x}$

$\Rightarrow f(x) = 4 - Ce^{-x}$.

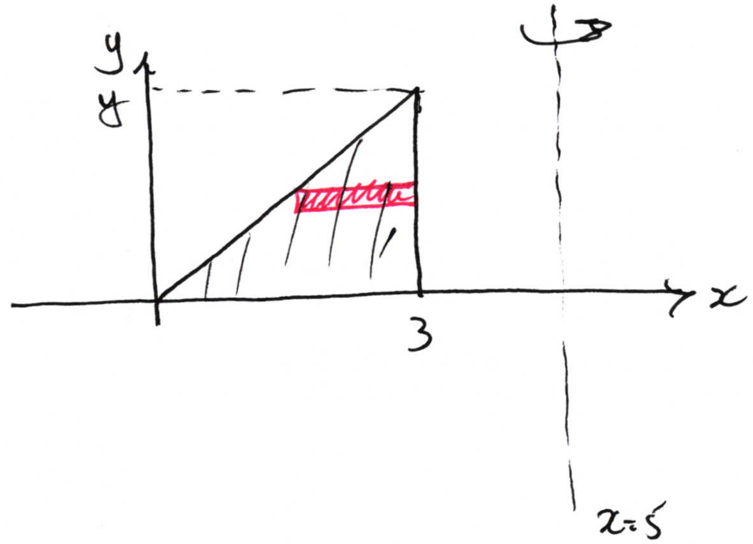
What is C? Plug $x=0$ into the original equation ~~to~~ get

$f(0) = 4\cdot 0 - \int_0^0 f(t)\,dt = 0$

$\Rightarrow 4 - C = 0 \Rightarrow C = 4$.

Hence, $\boxed{f(x) = 4 - 4e^{-x}}$

[6] **4.** The triangular region bounded by $y = x$, $y = 0$, and $x = 3$ is rotated about the line $x = 5$. Find the volume of the solid so generated.



Set this up as an integration in y.

$$V = \int_0^3 \left[\pi(5-y)^2 - \pi(5-3)^2\right] dy$$

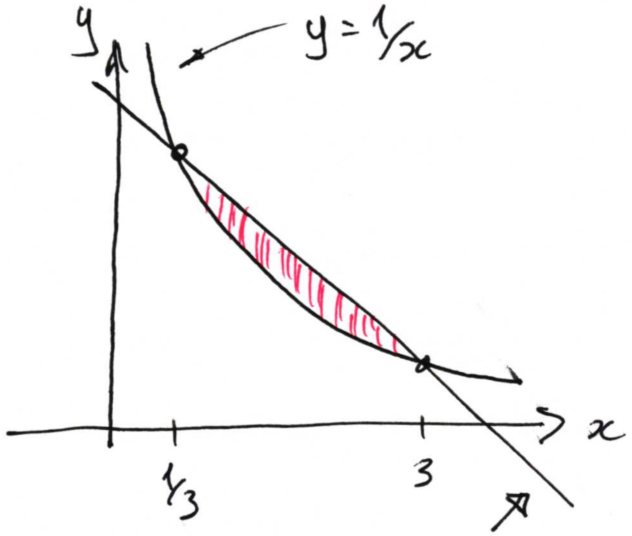
$$= \left[-\frac{\pi(5-y)^3}{3} - 4\pi y\right]_0^3$$

$$= -\frac{8}{3}\pi + \frac{125}{3}\pi - 12\pi$$

$$= 39\pi - 12\pi$$

$$= \boxed{27\pi}.$$

[6] **5.** Find the volume of the solid generated by rotating the finite region bounded by $y = 1/x$ and $3x + 3y = 10$ about the x-axis.



Intersection points:

$$3x + 3\left(\frac{1}{x}\right) = 10$$

$$\{x \neq 0\} \Rightarrow 3x^2 + 3 = 10x$$

$$\Rightarrow x = 1/3 \text{ OR } x = 3$$

$$3x + 3y = 10 \Rightarrow y = \frac{10}{3} - x$$

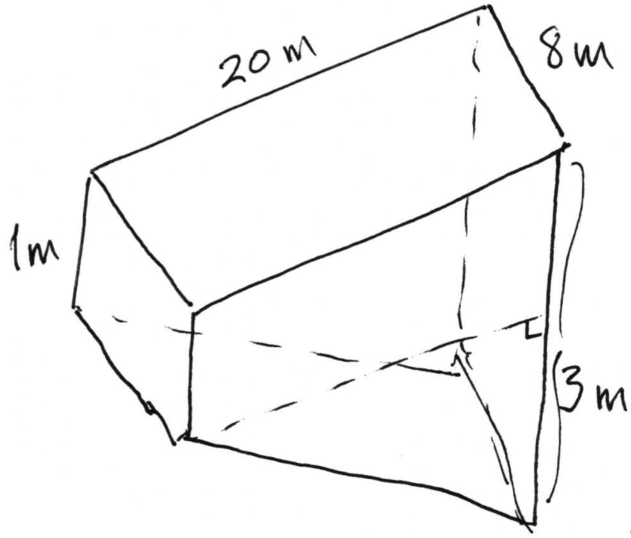
$$V = \int_{1/3}^{3} \left[\pi\left(\frac{10}{3} - x\right)^2 - \pi\left(\frac{1}{x}\right)^2 \right] dx$$

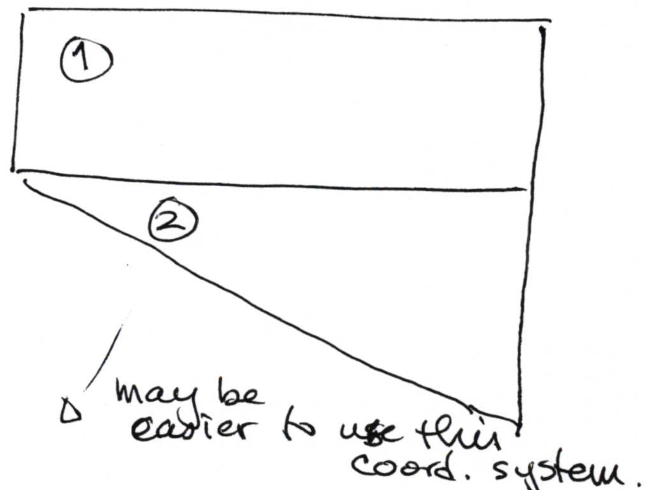
$$= \boxed{\frac{512\pi}{81}} .$$

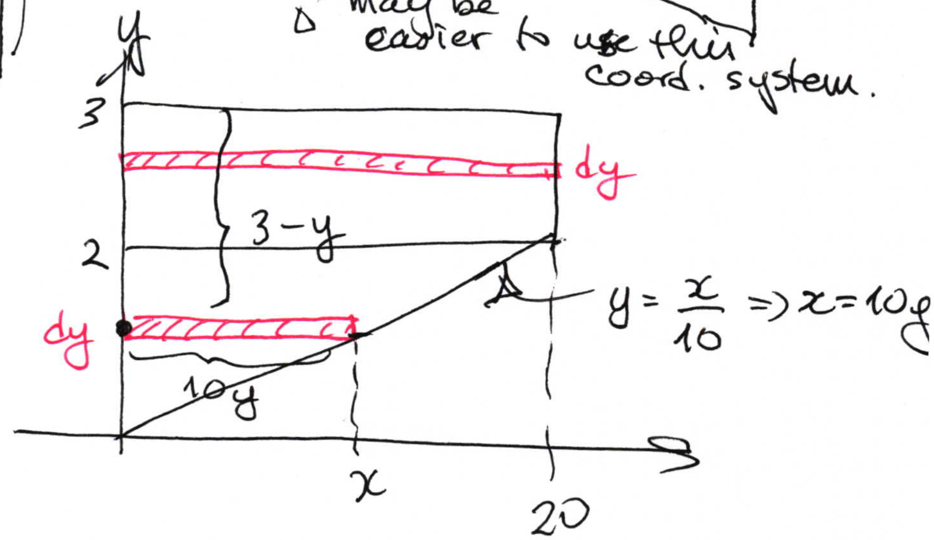
[6] **6.** A rectangular swimming pool 20 metres long and 8 metres wide has a sloping plane bottom so that the depth of the pool is 1 metre at one end and 3 metres at the other end. Find the total work that must be done to pump all the water in this pool when it is full over the top edge of the pool.

Let $g = 10\ m\,s^{-2}$ (You could use $9.8\ m\,s^{-2}$)

Split into two pieces







For ② $dW_{②} = 1000\frac{kg}{m^3}\cdot 10\,m\,s^{-2}\cdot \underbrace{8\cdot 10}_{\text{rectangular}}\,y\,(3-y)\,dy$

surface area depends on depth

$\Rightarrow W_2 = \int_0^2 10^5\cdot 8\cdot y(3-y)\,dy = \frac{8}{3}\times 10^6\ Nm$

constant surface area

For ① $dW_{①} = 1000\cdot 10\,(8\cdot 20)(3-y)\,dy$

$\Rightarrow W_{①} = \int_2^3 10^5\cdot 16\cdot(3-y)\,dy = \frac{16}{3}\times 10^6\ Nm$

Add these