Math 253, Section 102, Fall 2006
Practice Final Solutions

1. Determine whether the two lines L_1 and L_2 described below intersect. If yes, find the point of intersection. If not, say whether they are parallel or skew, and find the shortest distance between them.

   The line L_1 is described by the equations

$$x - 1 = 2y + 2, \quad z = 4,$$

   and the line L_2 passes through the points $P(2, 1, -3)$ and $Q(0, 8, 4)$.

   Solution. We first find two points on L_1, say $P_1(1, -1, 4)$ and $Q_1(3, 0, 4)$. The direction of L_1 is therefore parallel to $\mathbf{v}_1 = \vec{P_1Q_1} = (2, 1, 0)$. Similarly the direction of L_2 is parallel to $\mathbf{v}_2 = \vec{PQ} = (-2, 7, 7)$. Thus the two lines cannot be parallel. To determine whether they intersect we write down the parametric representation of L_1, namely

$$x = 2s + 3, \quad y = s, \quad z = 4.$$

   On the other hand, a parametric representation of of L_2 is

$$(x, y, z) = (2, 1, -1) + t\vec{PQ} = (2 - 2t, 1 + 7t, -3 + 7t).$$

   Setting $-3 + 7t = 4$ we obtain $t = 1$, while setting $1 + 7t = s$ gives $s = 8$. For these values of s and t, $x = 2 - 2t = 0$, and $x = 3 + 2s = 19$. Therefore the two lines do not intersect, i.e., they are skew.

   In order to find the distance between the lines, note that $\mathbf{c} = \vec{P_1P}$ is a connector between the lines and that $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} = (7, -14, 16)$ is normal to both lines. Therefore the distance between the lines is

$$D = \frac{|\mathbf{n} \cdot \mathbf{c}|}{|\mathbf{n}|} = \frac{133}{\sqrt{501}}.$$

□

2. Find and sketch the largest possible domain of the function

$$f(x, y, z) = \arcsin(3 - x^2 - y^2 - z^2).$$

Solution. The domain of the function is

$$\begin{aligned} D &= \left\{(x, y, z) : 0 \le 3 - x^2 - y^2 - z^2 \le 1\right\} \\ &= \left\{(x, y, z) : 2 \le x^2 + y^2 + z^2 \le 3\right\}. \end{aligned}$$

This describes a hollow spherical shell centered at the origin, whose inner radius is $\sqrt{2}$ and outer radius is $\sqrt{3}$. □

3. Find the equation of the tangent plane to the surface

$$yz = \ln(x + z)$$

at the point $(1, 0, 0)$.

Solution. We use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1, 0)$. Differentiating implicitly with respect to x gives

$$y\frac{\partial z}{\partial x} = \frac{1}{x+z}\left(1 + \frac{\partial z}{\partial x}\right), \quad \text{i.e.,} \quad \frac{\partial z}{\partial x}(1, 0, 0) = -1.$$

Similarly,

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\frac{\partial z}{\partial y}, \quad \text{i.e.,} \quad \frac{\partial z}{\partial y}(1, 0, 0) = 0.$$

The equation of the tangent plane is therefore

$$z = -(x - 1).$$

□

4. The plane $4x + 9y + z = 0$ intersects the elliptic paraboloid $z = 2x^2 + 3y^2$ in an ellipse. Find the highest and lowest points on this ellipse.

Solution. We have to optimize the function

$$f(x, y, z) = z$$

subject to the two constraints

$$g(x, y, z) = 4x + 9y + z = 0, \quad h(x, y, z) = 2x^2 + 3y^2 - z = 0.$$

Using the method of Lagrange multipliers, we set up the equations

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

which translate to

$$\begin{aligned} 0 &= 4\lambda + 4\mu x \\ 0 &= 9\lambda + 6\mu y, \text{ and} \\ 1 &= \lambda - \mu. \end{aligned}$$

Solve the three above equations together with the two constraints $g = 0$ and $h = 0$. Note first that $\mu \neq 0$, since if it were zero then λ would also be zero, contradicting the third equation above. Therefore

$$36\mu x = 24\mu y \quad \text{i.e.,} \quad y = \frac{3}{2}x.$$

Plugging this into $g = 0$ given

$$z = -4x - \frac{27}{2}x = -\frac{35}{2}x.$$

The constraint $h = 0$ then translates to

$$2x^2 + \frac{27}{4}x^2 + \frac{35}{2}x = 0, \qquad \text{or} \qquad \frac{35}{4}x(x + 2) = 0.$$

Thus we obtain two solutions for x, namely $x = 0$ and -2. For $x = 0$, $y = z = 0$, while for $x = -2$, $y = -3$ and $z = 35$. Therefore $(0, 0, 0)$ is the lowest point on the ellipse and and $(-2, -3, 35)$ is the highest. $\square$

5. Find the area that is cut from the surface $z = x^2 - y^2$ by the cylinder $x^2 + y^2 = 4$.

Proof. The projection of the surface described on the (x, y)-plane is the disk $x^2+y^2 \leq 4$. The surface area is obtained by integrating

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 4x^2 + 4y^2} \quad (\text{since } z = x^2 - y^2)$$

on this disk. Using cylindrical coordinates to simplify the resulting integral, we find the area to be

$$\begin{aligned} S &= \iint_{x^2+y^2 \leq 4} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx \\ &= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \sqrt{1 + 4r^2}\, r d\theta\, dr \\ &= \frac{\pi}{6}(17\sqrt{17} - 1). \end{aligned}$$

□

6. You are standing at the point where $x = y = 100$ feet on the side of a mountain whose height (in feet) above the sea level is given by

$$z = f(x, y) = \frac{1}{1000}(3x^2 - 5xy + y^2),$$

with the x-axis pointing east and the y-axis pointing north.

(a) If you head northeast, will you be ascending or descending? How fast?

Solution. Northeast is given by the direction $\mathbf{v} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$. The direction derivative along this direction at the point $x = y = 100$ is given by

$$\begin{aligned} D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f &= \mathbf{v} \cdot \frac{1}{1000}((6x - 5y)\mathbf{i} + (-5x + 2y)\mathbf{j}) \\ &= \frac{1}{1000\sqrt{2}}(x - 3y) = -\frac{1}{5\sqrt{2}}. \end{aligned}$$

I will therefore be descending at the rate of $\sqrt{2}/10$ feet per second if I head northeast. □

(b) In which direction should you head in order to descend the fastest?

Solution. The fastest increase happens along ∇f. The direction of fastest decrease is therefore

$$-\frac{\nabla f}{|\nabla f|} = -\frac{(6x-5y)\mathbf{i} + (-5x+2y)\mathbf{j}}{\sqrt{(6x-5y)^2 + (-5x+2y)^2}}$$
$$= -\frac{(100\mathbf{i} - 300\mathbf{j})}{100\sqrt{10}} = \frac{1}{\sqrt{10}}(-\mathbf{i} + 3\mathbf{j}).$$

□

(c) Suppose that you decide to move in a direction that makes an angle of 45° with $(1,-3)$. How fast will you be ascending or descending then?

Solution. Note from part (b) that $(1,-3)$ is the direction of ∇f. Therefore the direction derivative here is given by

$$\begin{aligned} D_{\mathbf{w}}f(100,100) &= \mathbf{w}\cdot\nabla f = |\nabla f|\cos 45^\circ \\ &= \frac{1}{1000}100\sqrt{10}\frac{1}{\sqrt{2}} = \frac{\sqrt{5}}{10}\text{feet per second.} \end{aligned}$$

□

(d) In which direction should you be moving in order to remain at the same altitude?

Solution. We need to move perpendicular to ∇f, i.e., along the direction

$$\pm\frac{1}{\sqrt{10}}(3\mathbf{i}+\mathbf{j}).$$

$\square$

7. Compute the value of the triple integral

$$\iiint_E z dV,$$

where E is the region between the surfaces $z = y^2$ and $z = 8 - y^2$ for $-1 \le x \le 1$.

Solution. Both the surfaces $z = y^2$ and $z = 8 - y^2$ represent parabolic cylinders. They intersect along the two lines $(x, \pm, 4)$. The region E therefore is bounded by $z = 8-y^2$ on the top, $z = y^2$ on the bottom, and its projection onto the xy-plane is the rectangle $[-1, 1] \times [-12]$ (Draw a picture to verify these statements). The value of the triple integral is therefore

$$\begin{aligned} I &= \int_{-1}^{1} \int_{-2}^{2} \int_{y^2}^{8-y^2} z \, dz \, dy \, dx \\ &= \int_{-1}^{1} \int_{-2}^{2} (32 - 8y^2) \, dy \, dx \\ &= \int_{-1}^{1} \frac{256}{3} \, dx = \frac{512}{3}. \end{aligned}$$

□

8. According to van der Waal's equation, 1 mol of gas satisfies the equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = cT$$

where p, V and T denote pressure (in atm), volume (in cm^3) and temperature (in kelvins) respectively, and a, b, c are constants. Suppose there exists a gas of volume 2 cm^3, pressure 1 atm and at temperature 5K for which $a = 16$, $b = c = 1$. Use differentials to approximate the change in its volume if p is increased 2 atm and T is increased to 8K.

Solution. We differentiate van der Waal's equation implicitly with respect to p and T to determine V_p and V_T respectively. Check that these are

$$V_p = \frac{V^3(V - b)}{aV - 2ab - pV^3} = -1, \quad V_T = \frac{cV^3}{pV^3 - aV + 2ab} = 1.$$

Now we use the linear approximation $dV = V_T dT + V_p dp$ to get $dV = 3 - 1 = 2$ cm^3. □

9. Evaluate

$$\iiint_E xyz\, dV$$

where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$. Here ρ and ϕ have the same interpretation as in spherical coordinates.

Solution.

$$\begin{aligned}\iiint_E xyz\, dV &= \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_2^4 (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi) \\ &\qquad\qquad \times \rho^2 \sin\phi d\rho d\theta d\phi \\ &= \int_0^{\frac{\pi}{3}} \sin^3\phi \cos\phi d\phi \int_0^{2\pi} \sin\theta\cos\theta d\theta \int_2^4 \rho^5 d\rho \\ &= 0,\end{aligned}$$

since the second integral (in θ) is zero. □

10. Identify all the local maximum, minimum and saddle points of the function

$$f(x, y) = (x^2 + y)e^{\frac{y}{2}}.$$

Proof. We find that $f_x = 2xe^{\frac{y}{2}}$ and $f_y = e^{\frac{y}{2}}(2+x^2+y)$. Therefore the only critical point of the function is $(0, -2)$. Since $D(0, -2) = e^{-2} > 0$ and $f_{xx}(0, -2) > 0$, by the second derivative test, $f(0, -2) = -2/e$ is a local minimum. $\square$