Math 253, Section 102, Fall 2006
Practice Midterm Solutions

Name: **SID:**

Instructions

- The total time is 50 minutes.
- The total score is 100 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators and cheat sheets are not allowed.

Problem	**Points**	**Score**
1	15	
2	15	
3	10	
4	10	
5	10	
6	20	
7	20	
TOTAL	25	

1. Prove that the lines

$$x - 1 = \frac{1}{2}(y+1) = z - 2 \quad \text{ and } x - 2 = \frac{1}{3}(y-2) = \frac{1}{2}(z-4)$$

intersect. Find an equation of the only plane that contains them both.

(7 + 8 = 15 points)

Solution. Let us first rewrite the equations of the lines in parametric form :

$$x - 1 = \frac{y+1}{2} = z - 2 = t, \qquad \text{or} \qquad \begin{cases} x &= 1 + t, \\ y &= 2t - 1, \\ z &= 2 + t, \end{cases}$$

and similarly,

$$x - 2 = \frac{y-2}{3} = \frac{z-4}{2} = s, \qquad \text{or} \qquad \begin{cases} x &= 2 + s, \\ y &= 3s + 2, \\ z &= 2s + 4. \end{cases}$$

The two lines will intersect if and only if the following system of equations

$$\begin{aligned} 1 + t &= 2 + s \\ 2t - 1 &= 3s + 2 \\ 2 + t &= 2s + 4. \end{aligned}$$

has a solution. Solving the first two equations in t and s, we obtain $t = 0, s = -1$, which also solves the third equation. Therefore the two lines intersect at the point $(1, -1, 2)$.

The two lines are parallel to $\mathbf{v_1} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v_2} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ respectively. Therefore the plane that contains both these lines

- has to pass through $(1, -1, 2)$ and
- has normal vector along the direction $\mathbf{v_1} \times \mathbf{v_2} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

The equation of the plane is therefore

$$(x - 1) - (y + 1) + (z - 2) = 0, \qquad \text{or} \qquad x - y + z = 4.$$

□

2. For each of the following, either compute the limit or show that the limit does not exist.

(7 + 8 = 15 points)

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)^{\frac{3}{2}}},$$

Solution. Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Note that $r \longrightarrow 0$ and $(x, y) \longrightarrow (0, 0)$. Thus,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)^{\frac{3}{2}}} &= \lim_{r\to 0} \frac{r^4(\cos^4\theta+\sin^4\theta)}{r^3} \\ &= \lim_{r\to 0} r(\cos^4\theta+\sin^4\theta) \\ &= 0.\end{aligned}$$

The last step follows by squeeze theorem, since $0 \leq \sin^4\theta + \cos^4\theta \leq 2$, and therefore,

$$0 \leq r(\cos^4\theta+\sin^4\theta) \leq 2r \longrightarrow 0.$$

□

(b)

$$\lim_{(x,y)\to(0,0)} \frac{2x^3y^2}{x^6+y^4}.$$

Solution. We will show that the limit does not exist by computing the limit along the family of paths $y = mx^{\frac{3}{2}}$, where m is an arbitrary parameter.

$$\begin{aligned}\lim_{\substack{y=mx^{\frac{3}{2}} \\ x\to 0}} \frac{2x^3y^2}{x^6+y^4} &= \lim_{x\to 0} \frac{2x^3m^2x^3}{x^6+m^4x^6} \\ &= \frac{2m^2}{m^4+1}.\end{aligned}$$

Since the limit depends on m, we obtain different limits along different paths. Therefore the limit does not exist. □

3. There is only one point at which the plane tangent to the surface

$$z = x^2 + 2xy + 2y^2 - 6x + 8y$$

is horizontal. Find it.

(10 points)

Solution. Let

$$f(x, y) = x^2 + 2xy + 2y^2 - 6x + 8y,$$

and let (x_0, y_0, z_0) be the point on the surface where the tangent plane is horizontal. This means that the normal direction to the plane is parallel to $\mathbf{k}$. Recall that the tangent plane at (x_0, y_0, z_0) is given by the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

therefore the normal direction to the plane lies along the direction $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$. Thus for the tangent plane to be horizontal, we must have

$$f_x(x_0, y_0) = 2x_0 + 2y_0 - 6 = 0,$$
$$f_y(x_0, y_0) = 2x_0 + 4y_0 + 8 = 0.$$

Solving the two equations we obtain $x_0 = 10$, $y_0 = -7$, $z_0 = -58$. $\square$

4. Identify the surface $x = \sin y$ in (x, y, z)-space, and sketch its graph.

(3 + 7 = 10 points)

Solution. The surface is a cylinder whose axis is the z-axis. Its projection on the (x, y)-plane is the curve $x = \sin y$. □

5. You buy a giftbox whose dimensions are 10 cm by 15 cm by 20 cm, but there may be a possible error of 0.1 cm in each. You want to buy just enough gift-wrapping paper to fully cover your box. What is the maximum error you should allow for while purchasing the paper?

(10 points)

Solution. Let x, y and z denote the length, width and height of the box respectively. The surface area is then given by

$$S = 2(xy + yz + zx).$$

The error incurred in computing the surface is given in terms of the differential

$$dS = 2(y + z)dx + 2(z + x)dy + 2(x + y)dz.$$

Substituting $x = 10$, $y = 15$, $z = 20$ and $dx = dy = dz = 0.1$, we obtain

$$dS = 18\text{cm}^2.$$

□

6. The sun is melting a rectangular block of ice. When the block's height is 1 m and the edge of its square base is 2 m, its height is decreasing at 20 cm/hr and its base edge is decreasing at 30 cm/hr. How fast is the volume of the ice block shrinking at that instant?

(20 points)

Solution. Let x denote the sidelength of the square base of the ice block, and let h denote the height. Then the volume is

$$V = x^2 h.$$

We need to compute $\frac{\partial V}{\partial t}$ when $x = 2$ m, $h = 1$ m, $\frac{\partial x}{\partial t} = -0.3$ m/hr and $\frac{\partial h}{\partial t} = -0.2$ m/hr. By the chain rule,

$$\begin{aligned} \frac{\partial V}{\partial t} &= 2xh\frac{\partial x}{\partial t} + x^2\frac{\partial h}{\partial t} \\ &= -2\text{m}^3/\text{hr}. \end{aligned}$$

□

7. Suppose that $w = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

(20 points)

Solution. By the chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial r} &= f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} \\
&= f_x\cos\theta + f_y\sin\theta, \\
\frac{\partial w}{\partial \theta} &= f_x\frac{\partial x}{\partial \theta} + f_y\frac{\partial y}{\partial \theta} \\
&= -f_x r\sin\theta + f_y r\cos\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 &= (f_x\cos\theta + f_y\sin\theta)^2 + (-f_x\sin\theta + f_y\cos\theta)^2 \\
&= \left[f_x^2\cos^2\theta + f_y^2\sin^2\theta + 2f_xf_y\sin\theta\cos\theta\right] + \\
&\qquad \left[f_x^2\sin^2\theta + f_y^2\cos^2\theta - 2f_xf_y\sin\theta\cos\theta\right] \\
&= f_x^2(\cos^2\theta + \sin^2\theta) + f_y^2(\sin^2\theta + \cos^2\theta) \\
&= f_x^2 + f_y^2 \\
&= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2.
\end{aligned}$$

□