Math 253, Section 102, Fall 2006

SAMPLE PROBLEMS FROM WEEK 2

Example 1 : Find the angle θ between the planes P_1 and P_2 with equations

$$2x + 3y - z = -3 \quad \text{and} \quad 4x + 5y + z = 1 \text{ respectively.}$$

Then write the equation of their line of intersection L in symmetric form.

Solution. The angle between two planes is the angle between their normals. The normal vectors to P_1 and P_2 are given by $\mathbf{n}_1 = (2, 3, -1)$ and $\mathbf{n}_2 = (4, 5, 1)$ respectively. Therefore

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{22}{\sqrt{14}\sqrt{42}}.$$

Hence $\theta = \cos^{-1}(\frac{11}{21}\sqrt{3}) \approx 24.87°$.

To find the equation of L, we need to find a point on L and its direction. To find a point, we can substitute an arbitrarily chosen value of x into the equations of the given planes and then solve the resulting equations for y and z. With $x = 1$ we get the equations

$$2 + 3y - z = -3$$
$$4 + 5y + z = 1.$$

The common solution is $y = -1$, $z = 2$. Thus the point $(1, , -1, 2)$ lies on the line L. Next we need to find a vector $\mathbf{v}$ parallel to L. The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are both normal to L (which lies on both P_1 and P_2), therefore

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (8, -6, -2).$$

We therefore have the symmetric equation

$$\frac{x-1}{8} = \frac{y+1}{-6} = \frac{z-2}{-2}.$$

□

Example 2 : Prove that the lines L_1 and L_2 given by

$$x - 1 = \frac{(y+1)}{2} = z - 2 \quad \text{and} \quad x - 2 = \frac{(y-2)}{3} = \frac{(z-4)}{2}$$

intersect. Find an equation of the only plane that contains them both.

Solution. Solving the equation of the two lines, we find that $P(1,-1,2)$ is the unique point that lies on both lines. We now need to find the normal direction to the plane that contain L_1 and L_2. Since L_1 and L_2 have directions parallel to $\mathbf{v}_1 = (1,2,1)$ and $\mathbf{v}_2 = (1,3,2)$ respectively, the direction $\mathbf{n}$ perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, and therefore,

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (1,-1,1).$$

The equation of the plane containing L_1 and L_2 is therefore

$$1(x-1) - y(y+1) + 2(z-1) = 0, \qquad \text{or } x - y + z = 4.$$

□

Example 3 : A child pulls a rope attached to a sled along the ground. The rope is inclined at an angle of 30° from the ground. If the child exerts a constant force of 20 lb, how much work is done in pulling the sled a distance of one mile?

Solution. We are given that $|\mathbf{F}| = 20(\text{lb})$ and $|\mathbf{D}| = 5280(\text{ft})$, where $\mathbf{F}$ and $\mathbf{D}$ denote force and displacement respectively. The work is given by

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 30^\circ = (20)(5280)\frac{\sqrt{3}}{2} \approx 91452(ft.lb).$$

□

Example 4 : Show that the points $A(1,-1,2)$, $B(2,0,1)$, $C(3,2,0)$ and $D(5,4,-2)$ lie on the same plane.

Solution. Recall that the scalar triple product $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ equals the volume of a parallelepiped with edges given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar (i.e., lie on the same plane), then the parallelepiped they generate has volume zero.

Therefore, in order to show that A, B, C and D lie on the same plane, it suffices to show that $|\vec{AB}\cdot(\vec{AC}\times\vec{AD})| = 0$. We compute

$$\vec{AB} = (1,1,-1),\ \vec{AC} = (2,3,-1) \ \text{ and } \ \vec{AD} = (4,5,-1),$$

which yields

$$|\vec{AB}\cdot(\vec{AC}\times\vec{AD})| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 4 & 5 & -4 \end{vmatrix} = 1.(-2) - 1.0 + (-1).(-2) = 0.$$

□

Example 5 : Find the shortest distance between points on the line L_1 with paramtric equations

$$x = 7 + 2t, \quad y = 11 - 5t, \quad z = 13 + 6t,$$

and the line L_2 on intersection of the planes $3x - 2y + 4z = 10$ and $5x + 3y - 2z = 15$.

Solution. First check that the two lines are skew, i.e., they neither intersect nor are they parallel (we worked out a problem like this in class). So it makes sense to find the distance between them. In order to find the shortest distance between L_1 and L_2, we need to find two points P_1 and P_2 on L_1 and L_2 respectively, then project $\vec{P_1P_2}$ in the direction $\mathbf{n}$ perpendicular to both L_1 and L_2. In other words,

$$\text{distance between } L_1 \text{ and } L_2 = \text{comp}_{\mathbf{n}}(\vec{P_1P_2}).$$

There can be many choices for P_1, but choosing $t = 0$ we obtain $P_1 = (7, 11, 13)$. The direction of L_1 is given by $\mathbf{v}_1 = (2, -5, 6)$. Similarly, setting $x = 4$ in the two equations for L_2 we obtain $P_2 = (4, -3, -2)$. Check using your favorite method that the direction of L_2 is parallel to $\mathbf{v}_2 = (-8, 26, 19)$. Therefore $\vec{P_1P_2} = (-3, -14, -15)$, and $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-251, -86, 12)$. Hence the distance between the two lines is

$$D = \frac{|\vec{P_1P_2} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{1777}{\sqrt{70541}}.$$

□