Math 253, Section 102, Fall 2006

Sample Problems from Week 3

Example 1 : Describe and sketch the graphs of the following equations. (i) $y^2+4x^2-9z^2=36$, (ii) $y=4x^2+9z^2$, (iii) $z=2y^2-z^2$.

Solution. We will only specify the type of the surface; sketching the graph is left to the student.
(i) The graph is a hyperboloid of one sheet with axis the z-axis. Please normal to the z-axis meet the surface in ellipses. Planes containing the z-axis meet it in both branches of a hyperbola.
(ii) The graph is an elliptic paraboloid opening in the positive y-direction, with axis the nonnegative y-axis and vertex at the origin.
(iii) Complete the square in z to obatin

$$\begin{aligned} 2y^2 - z^2 - z &= 0; \\ 2y^2 - z^2 - z - \frac{1}{4} &= -\frac{1}{4}; \\ 2y^2 - \left(z + \frac{1}{2}\right)^2 &= -\frac{1}{4}. \end{aligned}$$

Since x is missing from the equation, the graph has to be a cylinder. It meets the yz-plane in the hyperbola with equation

$$\left(z + \frac{1}{2}\right)^2 - 2y^2 = \frac{1}{4}.$$

Thus the graph is a hyperbolic cylinder parallel to the x-axis. □

Example 2 : Prove that the projection into the xz-plane of the intersection of the paraboloids $y=2x^2+3z^2$ and $y=5-3x^2-2z^2$ lies in a circle.

Solution. The intersection of the paraboloids satisfies both defining equations, and therefore lies on the surface with equation

$$2x^2 + 3z^2 = 5 - 3x^2 - 2z^2, \quad \text{or } x^2 + z^2 = 1.$$

This surface is a circular cylinder normal to the xz-plane. Therefore the projection of the intersection of the two paraboloids into the xz-plane lies on the circle given by the equations

$$y = 0, \quad x^2 + z^2 = 1.$$

□

Example 3 : Describe the graphs of the given equations. (It is understood that equations involving r are in cylindrical coordinates and those including ρ or ϕ are in spherical coordinates.)

$$\text{(i) } \rho^2 - 4\rho + 3 = 0, \quad \text{(ii) } z^2 = r^4.$$

Solution. (i) The graph of the spherical equation can be written in the form

$$(\rho - 1)(\rho - 3) = 0,$$

so that $\rho = 1$ or $\rho = 3$. The graph consists of all points that satisfy either of these two equations. Hence the graph consists of two concentric spherical surfaces, both centered at the origin, and of radii 1 and 3 respectively. (ii) The cylindrical equation $z^2 = r^4$ can be rewritten as $z = \pm r^2$, which can be expressed in Cartesian form as

$$z = x^2 + y^2 \quad \text{or } z = -(x^2 + y^2).$$

The graph thus consists of all points that satisfy either of the last two equations, hence it consists of two circular paraboloids, each with axis the z-axis, vertex at the origin; one opens upward and the other opens downward. □

Example 4 : Convert the following equation both to cylindrical and to spherical coordinates.

$$z = x^2 - y^2.$$

Solution. The Cartesian equation above takes the cylindrical form

$$z = r^2 \cos^2\theta - r^2 \sin^2\theta = r^2 \cos 2\theta.$$

It takes the spherical form

$$\begin{aligned}
\rho\cos\phi &= \rho^2 \sin^2\phi \cos^2\theta - \rho^2 \sin^2\phi \sin^2\theta, \\
\rho\cos\phi &= (\rho\sin\phi)^2(\cos^2\theta - \sin^2\theta), \\
\rho\cos\phi &= (\rho\sin\phi)^2 \cos 2\theta, \\
\cos\phi &= \rho \sin^2\phi \cos 2\theta.
\end{aligned}$$

□

Example 5 : State the largest possible domain of the given functions.

$$\text{(i) } f(x,y) = \sqrt{2x} + (3y)^{\frac{1}{3}}, \quad \text{(ii) } f(x,y) = \sin^{-1}(x^2 + y^2).$$

Solution. (i) Any real number has a unique real cube root, hence $(3y)^{\frac{1}{3}}$ is always well-defined. However $\sqrt{2x}$ is real if and only if $x \geq 0$. Therefore the domain of f consists of all points (x, y) such that $x \geq 0$, in other words, the right half plane.

(ii) Because $\arcsin z$ is a real number if and only if $-1 \leq z \leq 1$,, the domain of f consists of points (x, y) for which $x^2 + y^2 \leq 1$; that is, the set of all points on and within the unit curcle. □

Example 6 : Describe the level curve of the function

$$f(x, y) = e^{-x^2-y^2}.$$

Solution. Because $e^{-(x^2+y^2)}$ is constant exactly where x^2+y^2 is constant, the level curves of f are circles centered at the origin. □