# Math 253, Section 102, Fall 2006

## Sample Problems from Week 4

**Example 1 :** Determine whether the following limits exist. If yes, find the limit. If not, justify.

$$\text{(i)} \lim_{(x,y)\to(0,0)} \arctan\left(-\frac{1}{x^2+y^2}\right),$$

$$\text{(ii)} \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4},$$

$$\text{(iii)} \lim_{(x,y,z)\to(0,0,0)} \sin\left(\frac{1}{x^2+y^2+z^2}\right).$$

*Solution.* (i) Convert to polar coordinates; i.e., set $r = \sqrt{x^2+y^2}$. Then $r \to 0$ as $(x,y) \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \arctan\left(-\frac{1}{x^2+y^2}\right) = \lim_{r\to 0} \arctan\left(-\frac{1}{r^2}\right) = \lim_{z\to-\infty} \arctan z = -\frac{\pi}{2},$$

where at the last but one step we have substituted $z = -1/r^2$.

(ii) The substitution $y = mx$ yields

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4} = \lim_{x\to 0} \frac{x^4(1-m^4)}{x^4(1+m^2+m^4)} = \frac{1-m^4}{1+m^2+m^4}.$$

Hence if $(x,y) \to (0,0)$ along the line $y = 0$ (where $m = 0$, then the limit is 1, whereas if $(x,y) \to (0,0)$ along the line $y = x$ (when $m = 1$), the limit is 0. Therefore the limit does not exist.

(iii) Using spherical coordinates yields

$$\lim_{(x,y,z)\to(0,0,0)} \sin\left(\frac{1}{x^2+y^2+z^2}\right) = \lim_{\rho\to 0} \sin\left(\frac{1}{\rho^2}\right) = \lim_{t\to 0+} \sin\left(\frac{1}{t}\right).$$

Since the graph of $\sin(1/t)$ oscillates arbitrarily fast between -1 and 1 near $t = 0$, the limit does not exist. □

**Example 2 :** Determine the largest set of points in the $xy$-plane on which $f(x,y) = \tan(1/(x+y))$ defines a continuous function.

*Solution.* Because the tangent function is continuous on the set $\mathbb{R} \setminus \{\pm\pi/2, \pm3\pi/2, \pm5\pi/2\cdots\}$, the given function $f$ has discontinuities whenever

$$x+y=0 \quad \text{or} \quad \frac{1}{x+y} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots.$$

The set of discontinous points is therefore the union of an infinite number of parallel straight lines, given by

$$x + y = \pm\frac{2}{(2n+1)\pi}, \quad n = 0, 1, 2, \cdots \text{ and } x + y = 0.$$

□

**Example 3 :** Compute the first-order partial derivatives of the following function :

$$f(r, s, t) = (1 - r^2 - s^2 - t^2)e^{-rst}.$$

*Solution.*

$$\begin{aligned}
\frac{\partial f}{\partial r} &= -2re^{-rst} - st(1 - r^2 - s^2 - t^2)e^{-rst} \\
&= e^{-rst}(r^2st + s^3t + st^3 - 2r - st), \\
\frac{\partial f}{\partial s} &= -2se^{-rst} - rt(1 - r^2 - s^2 - t^2)e^{-rst} \\
&= e^{-rst}(rs^2t + r^3t + rt^3 - 2s - rt), \\
\frac{\partial f}{\partial t} &= -2te^{-rst} - rs(1 - r^2 - s^2 - t^2)e^{-rst} \\
&= e^{-rst}(rst^2 + r^3s + rs^3 - 2t - rs).
\end{aligned}$$

□

**Example 4 :** Describe the level surface of the fnction $f(x, y, z) = z + \sqrt{x^2 + y^2}$.

*Solution.* The level surface of $f$ is defined by the equation $f(x, y, z) = k$, where $k$ is a constant. This translates to $k - z = \sqrt{x^2 + y^2}$. The level surfaces of $f$ are therefore the lower nappes of circular cones with vertices on the $z$-axis. □

**Example 5 :** Discuss the continuity of the function

$$f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0. \end{cases}$$

*Solution.* The ratio of two continuous functions is always continuous, as long as the denominator does not vanish. Therefore $f$ is continuous at every $(a, b)$ such that $ab \neq 0$. We therefore only need to verify continuity at a point where $ab = 0$. Using the substitution $z = xy$ and the basic trigonometric limit $\sin t/t \to 1$ as $t \to 0$, we get

$$\lim_{(x,y)\to(a,b)} \frac{\sin(xy)}{xy} = \lim_{z\to ab} \frac{\sin z}{z} = 1 = f(a, b).$$

Therefore $f$ is continuous at all $(a, b) \in \mathbb{R}^2$. $\square$

**Example 6 :** Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$yz = \ln(x + z).$$

*Solution.* Differentiating the equation with respect to $x$ we get,

$$y\frac{\partial z}{\partial x} = \frac{1 + \frac{\partial z}{\partial x}}{x + z}.$$

Solving for $\partial z/\partial x$ gives

$$\frac{\partial z}{\partial x} = \frac{1}{y(x + z) - 1}.$$

Similarly differentiation with respect to $y$ yields

$$z + y\frac{\partial z}{\partial y} = \frac{\frac{\partial z}{\partial y}}{x + z},$$

from which we obtain

$$\frac{\partial z}{\partial y} = \frac{z(x + z)}{1 - y(x + z)}.$$

$\square$