Multivariable Calculus - Math 253, Section 102

Fall 2006

Solutions for Midterm Review Worksheet

1. If $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$, find $f_x(0,0)$.

(Ans. $f_x(0,0) = 1$.)

Solution. By the definition of partial derivative,

$$\begin{aligned} f_x(0,0) &= \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{(h^3 + 0)^{\frac{1}{3}}}{h} \\ &= \lim_{h \to 0} \frac{h}{h} = 1. \end{aligned}$$

□

2. For each of the following, determine whether the limit exists. If yes, compute the limit.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4 + y^2},$$

(Ans. limit is 0.)

(b)

$$\lim_{(x,y)\to(0,0)} \frac{y^2 + (1-\cos(2x))^2}{x^4 + y^2}.$$

(Ans. limit does not exist)

Solution. (a) We use the squeeze theorem to show that the limit exists. Notice that

$$0 \le \frac{y^2}{x^4 + y^2} \le 1.$$

Since $(1 - \cos(2x))$ is a nonnegative number, we can multiply all sides of the inequality by it without changing the order of the inequality. This gives

$$0 \le \frac{y^2(1 - \cos(2x))}{x^4 + y^2} \le (1 - \cos(2x)).$$

Both the left and right hand side approach 0 as $x \to 0$. Therefore by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{y^2(1 - \cos(2x))}{x^4 + y^2} = 0.$$

(b) We choose paths of the form $y = mx^2$ to show that the limit does not exist. On the path $y = mx^2$,

$$\begin{aligned} \frac{y^2 + (1 - \cos(2x))^2}{x^4 + y^2} &= \frac{m^2x^4 + (2\sin^2 x)^2}{x^4 + m^2x^4} \\ &= \frac{m^2 + \frac{4\sin^4 x}{x^4}}{1 + m^2}, \end{aligned}$$

which approaches $\frac{m^2+4}{1+m^2}$ as $x \to 0$. Since the limit along a path depends on m, an arbitrary parameter that depends on the path, the limit does not exist. □

3. (a) Identify the surface $x^2 - y^2 + 2z^2 = 1$.

(b) Find the point(s) on this surface where the direction perpendicular to the tangent plane is parallel to the line joining $(3, -1, 0)$ and $(5, 3, 6)$.

(Ans.$(\frac{\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}, \frac{\sqrt{6}}{2})$ and $(-\frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}, -\frac{\sqrt{6}}{2})$)

Proof. (a) The surface is an hyperboloid of one sheet with axis along the y-axis.

(b) Set

$$F(x, y, z) = x^2 - y^2 + 2z^2 - 1.$$

Let (x_0, y_0, z_0) be the point where the direction of the normal vector is parallel to $(2, 4, 6)$. The normal direction to the surface at (x_0, y_0, z_0) points along $(F_x, F_y, F_z) = (2x_0, -2y_0, 4z_0)$. Therefore, there exists a constant k such that

$$2x_0 = 2k, \quad -2y_0 = 4k, \quad 4z_0 = 6k.$$

Plugging this into the equation of the surface gives $k = \pm\sqrt{6}/3$, from which we get the coordinates of the point

$$(x_0, y_0, z_0) = k(1, -2, \frac{3}{2}) = \left(\frac{\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}, \frac{\sqrt{6}}{2}\right)$$
$$\text{and } \left(-\frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}\right).$$

□

4. You are standing at the point $(30, 20, 5)$ on a hill with the shape of the surface

$$z = \frac{1}{1000} \exp\left(-\frac{x^2 + 3y^2}{700}\right).$$

(a) In what direction should you proceed in order to climb most steeply?

(Ans. $\langle -\frac{60}{7}, -\frac{120}{7} \rangle$)

(b) At what angle from the horizontal will you initially be climbing in this case?

(Ans. $\arctan\left(\frac{60\sqrt{5}e^{-3}}{70000}\right)$)

(c) If instead of climbing as in part (a), you head directly west, what is your initial rate of ascent? At what angle to the horizontal will you be climbing initially?

(Ans. $\frac{6e^{-3}}{70000}$, $\arctan(6e^{-3}/70000)$)

Solution. (a) Set

$$f(x, y) = \frac{1}{1000} \exp\left(-\frac{x^2 + 3y^2}{700}\right).$$

Steepest ascent will be in the direction $\mathbf{u}$ along which the directional derivative $D_{\mathbf{u}}f$ will be maximized. We know that this will be in the direction

$$\mathbf{u} = \frac{\nabla f(30, 20)}{|\nabla f(30, 20)|}.$$

Now,

$$\nabla f(x, y) = \frac{1}{1000} \exp\left(-\frac{x^2 + 3y^2}{700}\right)\left(-\frac{2x}{700}, \frac{6y}{700}\right),$$

from which we get the direction to be

$$\mathbf{u} = (-1, -2)/\sqrt{5}.$$

Note that this has the same direction as the given answer.

(b) When $\mathbf{u}$ is as in part (a),

$$D_{\mathbf{u}} f = |\nabla f| = \frac{6}{70000} e^{-3}\sqrt{5}.$$

If θ is the angle that the initial climbing direction makes with the horizontal, then $\tan\theta = D_{\mathbf{u}} f$. The angle to the horizontal made while climbing the slope of steepest ascent is therefore

$$\theta = \arctan|\nabla f| = \arctan\left(\frac{6}{70000} e^{-3}\sqrt{5}\right).$$

Remark : Note that there was a typo in the answer given in the original worksheet.

(c) Same as parts (a) and (b) but with $\mathbf{u} = (-1, 0)$. □

5. Suppose that $w = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

Solution. We know that

$$\begin{cases} \frac{\partial x}{\partial r} &= \cos\theta \\ \frac{\partial y}{\partial r} &= \sin\theta \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial \theta} &= r\cos\theta. \end{cases}$$

By the chain rule we obtain,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = -r\frac{\partial w}{\partial x}\sin\theta + r\frac{\partial w}{\partial y}\cos\theta$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2} &= \frac{\partial}{\partial r}\left[\frac{\partial w}{\partial x}\right]\cos\theta + \frac{\partial}{\partial r}\left[\frac{\partial w}{\partial y}\right]\sin\theta \\
&= \cos\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial r}\right] + \\
&\qquad \sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial r}\right] \\
&= \cos\theta\left[\frac{\partial^2 w}{\partial x^2}\cos\theta + \frac{\partial^2 w}{\partial x\partial y}\sin\theta\right] + \\
&\qquad \sin\theta\left[\frac{\partial^2 w}{\partial x\partial y}\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin\theta\right] \\
&= \cos^2\theta\frac{\partial^2 w}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 w}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 w}{\partial \theta^2}.
\end{aligned}$$

In order to compute $\partial^2 w/\partial\theta^2$, we need to use product rule in conjunction with chain rule.

$$\begin{aligned}
\frac{\partial^2 w}{\partial \theta^2} &= -r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial x}\right) \\
&\qquad - r\sin\theta\frac{\partial w}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial y}\right) \\
&= -r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial w}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
&\quad - r\sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial \theta}\right]\\
&\quad + r\cos\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial \theta}\right]\\
&= -r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial w}{\partial y}\\
&\quad - r^2\sin\theta\left[-\sin\theta\frac{\partial^2 w}{\partial x^2}+\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad + r^2\cos\theta\left[-\sin\theta\frac{\partial^2 w}{\partial x\partial y}+\cos\theta\frac{\partial^2 w}{\partial y^2}\right]\\
&= -r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial w}{\partial y}\\
&\quad + r^2\sin^2\theta\frac{\partial^2 w}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 w}{\partial y^2}.
\end{aligned}
$$

Now use these to show that the right hand side of the identity given in the problem equals the left hand side. □

6. A rectangular block has dimensions $x = 3m$, $y = 2m$ and $z = 1m$. If x and y are increasing at 1 cm/min and 2 cm/min respectively, while z is decreasing at 2 cm/min, are the block's volume and surface area increasing or decreasing? At what rates?

(Ans. volume of box decreases at the rate of 40,000 cm^3/min; surface area increases at the rate of 200 cm^2/min.)

Solution. Let V and S denote the volume and surface area of the rectangular box respectively. Then

$$V = xyz \quad \text{and } S = 2(xy + yz + zx).$$

Then,

$$\begin{aligned} dV &= yzdx + zxdy + xydz \\ &= (200)(100)(1) + (300)(100)(2) + (300)(200)(-2) \\ &= -40000\text{cm}^3/\text{min}, \text{ and} \\ dS &= 2(y + z)dx + 2(z + x)dy + 2(x + y)dz \\ &= 2(300)(1) + 2(400)(2) + 2(500)(-2) \\ &= 200\text{cm}^2/\text{min}. \end{aligned}$$

□