Weighted Inequalities for Real-Analytic Functions in $\mathbb{R}^2$

Malabika Pramanik

November 5, 2004

Abstract

Let f and g be real-analytic functions near the origin in $\mathbb{R}^2$. Given $1 < p < \infty$, we obtain a characterization of the set of positive numbers ϵ and δ that ensures

$$\frac{|g|^\epsilon}{|f|^\delta} \in A_p(K)$$

for some small neighborhood K of the origin. A notion of stability is introduced in relation to A_p weights and a counterexample is presented to show that the two-dimensional weighted problem, unlike its analogue in dimension one, is not stable.

This paper contains the second in a list of results on weighted integrals in $\mathbb{R}^2$ and is a follow-up of [Pra]. The notation introduced in section 2 of [Pra] will be used throughout the discussion.

In this paper we investigate a certain kind of stability property of weighted integrals in $\mathbb{R}^2$ of the form

$$\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx, \tag{0.1}$$

where f and g are real-analytic, complex-valued functions in a neighborhood of the origin, $f(0) = g(0) = 0$ and ϵ and δ are positive numbers. When $g \equiv 1$, integrals of the above form arise naturally in numerous problems in harmonic analysis, specifically in connection with growth rate of real-analytic functions and decay rates of oscillatory integrals. In particular, for $g \equiv 1$ and in dimension $n = 2$, the problem of finiteness and stability (with respect to real-analytic perturbations of f) of the integral in (0.1) has been fully treated by Phong, Stein and Sturm [PSS99], while the related problems of determining the oscillation index of a two-dimensional oscillatory integral and stability thereof dates back to Varchenko [Var76] and Karpushkin [Kar86b] [Kar86a]. It would be of considerable interest, however, to tackle the issue in higher dimensions (i.e. $n \geqslant 3$) where the problem of finiteness and stability of the unweighted integral ((0.1) with $g \equiv 1$) is still poorly understood and finiteness and stability results are, at best, partial (see [Var76]).

Weighted integrals of the form (0.1) sometimes arise from their unweighted higher dimensional analogues after a suitable change of coordinates, especially if the higher dimensional f comes equipped with certain symmetries that can be exploited to reduce the dimension. A case in point is the important counterexample given by Varchenko in the context of oscillatory integrals in $\mathbb{R}^3$ (see section 5, [Var76]). This is a continuous real-analytic deformation of a function of three variables, where the oscillation index is known to possess a discontinuity. In our situation, Varchenko's example translates to

$$\iiint_{B^3 \subset \mathbb{R}^3} \frac{dx_1\, dx_2\, dx_3}{\left|(\lambda x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p}\right|^\delta}, \tag{0.2}$$

where λ is a real parameter and p is a sufficiently large natural number. Now, a few trivial size estimates coupled with a cylindrical change of coordinates transforms the above three-dimensional unweighted integral to a two-dimensional weighted one, given by

$$\iint_{B^2 \subset \mathbb{R}^2} \frac{|y|\, dy\, dx}{|(y^2 + x^4 + \lambda x^2)^2 + x^{4p} + y^{4p}|^\delta}. \tag{0.3}$$

Clearly, the integral in (0.2) converges if and only if the integral in (0.3) does. In general, the hope is that results for integrals of the form (0.1) would shed some light on the behavior of the higher-dimensional unweighted ones they arise from.

While uniform estimates on integrals of the form (0.1) are of interest, there are many ways in which such a stability problem may be formulated. One of them is to consider an integral of the form (0.1) in the setting of weighted inequalities. This is the approach we have chosen in the present paper. Following standard literature, we review the definition of A_p weights:

Definition 0.1 *Let $1 < p < \infty$. A locally integrable function $\omega : \mathbb{R}^n \to \mathbb{R}$*

is said to be an A_p weight (or satisfy the A_p inequality) on $\mathbb{R}^n$ if

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} \le A < \infty \tag{0.4}$$

for all balls $B \subset \mathbb{R}^n$. Here p' is the dual of p, i.e. $p^{-1} + p'^{-1} = 1$.
ω will be called an A_p weight on $V \subseteq \mathbb{R}^n$ if the uniform bound given by (0.4) holds for all balls $B \subset V$.

The theory of weighted inequalities has a rich history in the literature. For a discussion of the main results on this topic and further references, see for example chapter 5 of [Ste93].

The goal of this paper is to obtain for a fixed p with $1 < p < \infty$ a necessary and sufficient condition for the function

$$\frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}$$

to be an A_p weight in a small neighborhood of the origin.

The paper is divided into three sections. The first section is a review of the notation and terminology in [Pra] that will be needed for our analysis. The second section contains the statement and proof of the main result. Here we have followed the techniques used by Phong and Stein [PS97] to analyze a related problem, namely that of computing the decay rate of a degenerate oscillatory integral operator on $\mathbb{R}$. The final section provides an interpretation of the algebraic inequalities obtained in the statement of the main theorem. It also offers a comparison of the condition obtained with its analogue in dimension one.

1 Notation and Preliminary Reductions

We begin with a brief review of the notation introduced in [Pra]. In view of the Weierstrass Preparation Theorem, f and g may be expressed, after a nonsingular change of coordinates, as polynomials in y with coefficients in x, modulo some nonvanishing factors. Factoring out these nonvanishing terms we write f and g as

$$f(x,y) = x^{\tilde{\alpha}_1} y^{\tilde{\beta}_1} \prod_{\nu\,;\,\nu\in I_f} (y - r_\nu(x))\,, \quad g(x,y) = x^{\tilde{\alpha}_2} y^{\tilde{\beta}_2} \prod_{\mu\,;\,\mu\in I_g} (y - s_\mu(x))\,, \tag{1.1}$$

where $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1$ and $\tilde{\beta}_2$ are non-negative integers and $r_\nu(x)$, $s_\mu(x)$ are the non-trivial zeros of f and g respectively. I_f and I_g are index sets that are in one-to-one correspondence with the roots of f and g respectively. In a small neighborhood of the origin these roots admit fractional power series expansions in x, the so-called Puiseux series

$$r_\nu(x) = c_\nu x^{a_\nu} + O\left(x^{b_\nu}\right), \quad s_\mu(x) = c_\mu x^{a_\mu} + O\left(x^{b_\mu}\right).$$

Here the exponents $a_\nu, a_\mu, b_\nu, b_\mu$ are rational numbers and the leading coefficients c_ν, c_μ are nonzero scalars. We order the combined set of distinct leading exponents a_ν, a_μ-s into one increasing list of exponents a_l,

$$0 < a_1 < a_2 < \cdots < a_l < a_{l+1} < \cdots < a_N.$$

The *generalized multiplicity* of f (respectively g) corresponding to a_l, denoted by m_l (respectively n_l), is defined as follows

$$m_l := \#\{\nu : r_\nu(x) = c_\nu x^{a_l} + \cdots, \text{ for some } c_\nu \neq 0\},$$
$$n_l := \#\{\mu : s_\mu(x) = c_\mu x^{a_l} + \cdots, \text{ for some } c_\mu \neq 0\}.$$

If a_l does not occur as a leading exponent of any root of f (respectively g) we set $m_l = 0$ (respectively $n_l = 0$).

The following quantities arise naturally in the description of the Newton diagrams of f and g (for the definition of the Newton diagram see [PS97]):

$$A_l = \tilde{\alpha}_1 + a_1 m_1 + a_2 m_2 + \cdots + a_l m_l, \qquad B_l = \tilde{\beta}_1 + m_{l+1} + \cdots + m_N,$$
$$C_l = \tilde{\alpha}_2 + a_1 n_1 + a_2 n_2 + \cdots + a_l n_l, \qquad D_l = \tilde{\beta}_2 + n_{l+1} + \cdots + n_N,$$

and

$$\delta_l = \frac{1 + a_l}{A_l + a_l B_l}, \qquad \tilde{\delta}_l = \frac{1 + a_l}{C_l + a_l D_l}.$$

All of the above quantities are coordinate-dependent and may be defined in any coordinate system φ in which f and g have the representations given by (1.1). Such coordinate systems are called "good" and they have the following form :

$$(x, y) \mapsto (x, y - q(x)) \quad \text{or} \quad (x, y) \mapsto (x - q(y), y),$$

where q is any convergent real-valued Puiseux series in a neighborhood of the origin. For any good coordinate system φ, the *weighted Newton distance* of f and g associated to φ is defined as follows :

$$\delta_0(g, f, \epsilon; \varphi) := \min_l \left[\delta_l(\varphi)\left(1 + \frac{\epsilon}{\tilde{\delta}_l(\varphi)}\right)\right].$$

Here the index l runs through the combined set of leading exponents of Puiseux series of the roots of f and g expressed in the coordinate system φ ; $\delta_l(\varphi)$, $\tilde{\delta}_l(\varphi)$ are the values of δ_l and $\tilde{\delta}_l$ respectively computed in these coordinates. The weighted Newton distance plays an important role in the description of the set $S(g,f)$ given by

$$S(g,f) := \Big\{(\epsilon,\delta)\,;\,\epsilon,\delta>0,\ \int_{B(0;r)} \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}\,dy\,dx \;<\; \infty \\ \text{for all sufficiently small } r>0\Big\}.$$

For more details on the notation and a precise description of $S(g,f)$ in terms of the weighted Newton distance, see [Pra]. We shall use the properties of $S(g,f)$ outlined in the proof of Theorem 1 in [Pra] to solve the A_p weight problem mentioned above.

2 The Main Theorem

Theorem 2.1 *Suppose f and g are real-analytic (possibly complex-valued) functions in a neighborhood of the origin in $\mathbb{R}^2$, $f(0)=g(0)=0$. Let $1<p<\infty$. Then there exists a compact neighborhood K of the origin with*

$$\frac{|g|^{\epsilon}}{|f|^{\delta}} \in A_p(K)$$

if and only if both of the following conditions hold :

(1) $(\epsilon,\delta)\in S(g,f), \quad (\delta,\epsilon)\dfrac{p'}{p}\in S(f,g)$,

(2) *For every admissible transformation associated to the pair (f,g), and for every l,*

$$\left\{\begin{array}{ll} (a)\ -1<\epsilon D_{l-1}-\delta B_{l-1}<p-1 & \text{if } a_l>1 \\ (b)\ -1<\epsilon D_l-\delta B_l<p-1 & \text{if } a_l=1 \\ (c)\ -1<\epsilon C_l-\delta A_l<p-1 & \text{if } a_l<1 \end{array}\right\}$$

where a_l, A_l, B_l, C_l and D_l depend on the particular choice of coordinates mentioned above.

Proof of Theorem :

First we shall prove necessity. Note that we need $(\epsilon,\delta)\in S(g,f)$ and $\frac{p'}{p}(\delta,\epsilon)\in S(f,g)$ in order to ensure convergence of the integrals

$$\int_B \frac{|g|^\epsilon}{|f|^\delta}\,dy\,dx \quad \text{and} \quad \int_B \frac{|f|^{\delta\frac{p'}{p}}}{|g|^{\epsilon\frac{p'}{p}}}\,dy\,dx$$

respectively, for small balls B centered at the origin. Thus, one only needs to verify condition (2) in the statement of the theorem.

Let us first consider the admissible transformation $(x,y)\mapsto(x,y)$ and fix l such that $a_{l+1}>1$. Choose b such that $\max(1,a_l)<b<a_{l+1}$. Then it is a consequence of $|g|^\epsilon/|f|^\delta\in A_p(K)$ that

$$\sup_{\lambda<\lambda_0}\left(\frac{1}{\lambda^{2b}}\int_{B\left((\lambda,0);\lambda^b\right)}\frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\,dy\,dx\right)\left(\frac{1}{\lambda^{2b}}\int_{B\left((\lambda,0);\lambda^b\right)}\frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}}$$

$$=\sup_{\lambda<\lambda_0}\left(\int_{B(0;1)}\frac{|g(\lambda+x\lambda^b,y\lambda^b)|^\epsilon}{|f(\lambda+x\lambda^b,y\lambda^b)|^\delta}\,dy\,dx\right)\left(\int_{B(0;1)}\frac{|f(\lambda+x\lambda^b,y\lambda^b)|^{\delta\frac{p'}{p}}}{|g(\lambda+x\lambda^b,y\lambda^b)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}} \tag{2.1}$$

has to be finite for λ_0 sufficiently small.

Now, borrowing the notation from Theorem 1 of [Pra], we write

$$\begin{aligned} f(\lambda+x\lambda^b,y\lambda^b) &= (\lambda+x\lambda^b)^{\tilde\alpha_1}(y\lambda^b)^{\tilde\beta_1}\prod_{\nu\,;\,\nu\in I_f}(y\lambda^b-r_\nu(\lambda+x\lambda^b)) \\ &= (\lambda+x\lambda^b)^{\tilde\alpha_1}(y\lambda^b)^{\tilde\beta_1}\prod_{\nu\,;\,\nu\in I_f}(y\lambda^b-c_\nu(\lambda+x\lambda^b)^{a_\nu}-\cdots), \end{aligned}$$

where ν is an index that ranges over all the roots $y=r_\nu(x)$ of f in a neighborhood of the origin, a_ν is the leading exponent of r_ν and $c_\nu\neq 0$

is the leading coefficient. So,

$$\begin{aligned} f(\lambda + x\lambda^b, y\lambda^b) =& \lambda^{\tilde{\alpha}_1}(1 + x\lambda^{b-1})^{\tilde{\alpha}_1}\lambda^{b\tilde{\beta}_1}y^{\tilde{\beta}_1} \prod_{\nu\colon a_\nu \le a_l} \lambda^{a_\nu}\left(y\lambda^{b-a_\nu} - c_\nu(1 + x\lambda^{b-1})^{a_\nu} - \cdots\right) \\ & \times \prod_{\nu\colon a_\nu > a_l} \lambda^b\left(y - c_\nu\lambda^{a_\nu - b}(1 + x\lambda^{b-1})^{a_\nu} - \cdots\right) \\ =& \lambda^{A_l + bB_l}(1 + x\lambda^{b-1})^{\tilde{\alpha}_1}y^{\tilde{\beta}_1} \prod_{\nu\colon a_\nu \le a_l} \left(y\lambda^{b-a_\nu} - c_\nu(1 + x\lambda^{b-1})^{a_\nu} - \cdots\right) \\ & \times \prod_{\nu\colon a_\nu > a_l} \left(y - c_\nu\lambda^{a_\nu - b}(1 + x\lambda^{b-1})^{a_\nu} - \cdots\right). \end{aligned}$$

Since $b > 1$ and the Puiseux series $r_\nu(x)$ converge in a small neighborhood K of the origin, we have for every $(x, y) \in K$

$$\lim_{\lambda \to 0} \frac{f(\lambda + x\lambda^b, y\lambda^b)}{\lambda^{A_l + bB_l}} = C(f)y^{B_l},$$

where $C(f)$ is a a nonzero constant depending on f. Similarly,

$$\lim_{\lambda \to 0} \frac{g(\lambda + x\lambda^b, y\lambda^b)}{\lambda^{C_l + bD_l}} = C(g)y^{D_l}.$$

Now we can rewrite (2.1) as

$$\sup_{\lambda < \lambda_0} \left(\int_{B(0;1)} H_1^\lambda(x, y)\, dy\, dx\right)\left(\int_{B(0;1)} H_2^\lambda(x, y)\, dy\, dx\right)^{\frac{p'}{p}}, \tag{2.2}$$

where

$$\begin{aligned} H_1^\lambda(x, y) &:= \frac{|g(\lambda + x\lambda^b, y\lambda^b)|^\epsilon}{|f(\lambda + x\lambda^b, y\lambda^b)|^\delta}\lambda^{\delta(A_l + bB_l) - \epsilon(C_l + bD_l)}, \\ H_2^\lambda(x, y) &:= \frac{|f(\lambda + x\lambda^b, y\lambda^b)|^{\delta\frac{p'}{p}}}{|g(\lambda + x\lambda^b, y\lambda^b)|^{\epsilon\frac{p'}{p}}}\lambda^{\epsilon\frac{p'}{p}(C_l + bD_l) - \delta\frac{p'}{p}(A_l + bB_l)}, \end{aligned}$$

with

$$\begin{aligned} \lim_{\lambda \to 0} H_1^\lambda(x, y) &= h_1(x, y) := |y|^{D_l\epsilon - B_l\delta} \\ \lim_{\lambda \to 0} H_2^\lambda(x, y) &= h_2(x, y) := |y|^{B_l\delta\frac{p'}{p} - D_l\epsilon\frac{p'}{p}} \end{aligned}$$

for every (x, y) in K such that $y \neq 0$.
By Fatou's lemma,

$$\int_{B(0;1)} h_1(x,y)\,dy\,dx \leq \liminf_{\lambda\to 0} \int_{B(0;1)} H_1^{\lambda}(x,y)\,dy\,dx = \lim_{\lambda_n\to 0} \int_{B(0;1)} H_1^{\lambda_n}(x,y)\,dy\,dx$$

for some sequence $\{\lambda_n\}$, while

$$\int_{B(0;1)} h_2(x,y)\,dy\,dx \leq \liminf_{\lambda_n\to 0} \int_{B(0;1)} H_2^{\lambda}(x,y)\,dy\,dx = \lim_{\lambda_{n_k}\to 0} \int_{B(0;1)} H_1^{\lambda_{n_k}}(x,y)\,dy\,dx$$

for a suitable subsequence $\{\lambda_{n_k}\}$. Therefore,

$$\begin{aligned}
&\left(\int_{B(0;1)} h_1(x,y)\,dy\,dx\right)\left(\int_{B(0;1)} h_2(x,y)\,dy\,dx\right)^{\frac{p}{p'}} \\
&\leq \lim_{\lambda_{n_k}\to 0} \left(\int_{B(0;1)} H_1^{\lambda_{n_k}}(x,y)\,dy\,dx\right)\left(\int_{B(0;1)} H_2^{\lambda_{n_k}}(x,y)\,dy\,dx\right)^{\frac{p}{p'}} \\
&<\ \infty,
\end{aligned}$$

using (2.2) and the A_p condition. But this implies

$$\int_{B(0;1)} h_i(x,y)\,dy\,dx <\ \infty, \qquad i = 1,\ 2$$

i.e.,

$$-1 < \epsilon D_l - \delta B_l < p - 1$$

In fact, a closer look at the above arguments reveals that we have indeed proved both 2(a) and 2(b) in the statement of the theorem.

Next, let us consider the case when $a_l < 1$. Choose $b < 1$ such that $a_l < b < a_{l+1}$. The A_p condition implies that

$$\sup_{\lambda<\lambda_0} \left(\frac{1}{\lambda^{\frac{2}{b}}} \int_{B\left((0,\lambda);\lambda^{\frac{1}{b}}\right)} \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}\,dy\,dx\right)\left(\frac{1}{\lambda^{\frac{2}{b}}} \int_{B\left((0,\lambda);\lambda^{\frac{1}{b}}\right)} \frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}}$$

is finite for λ_0 small. As before, we compute

$$\begin{aligned} f(x\lambda^{\frac{1}{b}}, \lambda + y\lambda^{\frac{1}{b}}) &= (x\lambda^{\frac{1}{b}})^{\tilde{\alpha}_1}(\lambda + y\lambda^{\frac{1}{b}})^{\tilde{\beta}_1} \prod_{\nu} \Big((\lambda + y\lambda^{\frac{1}{b}}) - r_\nu(x\lambda^{\frac{1}{b}}) \Big) \\ &= \lambda^{\frac{\tilde{\alpha}_1}{b} + \tilde{\beta}_1} x^{\tilde{\alpha}_1} (1 + y\lambda^{\frac{1}{b}-1})^{\tilde{\beta}_1} \\ &\qquad \times \prod_{\nu :\, a_\nu < b} \lambda^{\frac{a_\nu}{b}} \Big(\lambda^{1-\frac{a_\nu}{b}} (1 + y\lambda^{\frac{1}{b}-1}) - c_\nu x^{a_\nu} - \cdots \Big) \\ &\qquad \times \prod_{\nu :\, a_\nu > b} \lambda \Big((1 + y\lambda^{\frac{1}{b}-1}) - c_\nu \lambda^{\frac{a_\nu}{b}-1} x^{a_\nu} - \cdots \Big), \end{aligned}$$

which leads to

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{\frac{A_l}{b} + B_l}} f(x\lambda^{\frac{1}{b}}, \lambda + y\lambda^{\frac{1}{b}}) = \tilde{C}(f) x^{A_l}$$

for every $(x, y) \in K$.The same argument involving Fatou's lemma given earlier now yields

$$-1 < \epsilon C_l - \delta A_l < p - 1.$$

It is important to observe that the above argument applies not only to real-analytic functions f and g, but to any function of the form

$$\prod_{\kappa} (y - t_\kappa(x)),$$

where the product above consists of a finite number of factors and each t_κ is a Puiseux series that converges in a small enough neighborhood of the origin.

Next, let us consider a general admissible transformation φ. Without loss of generality φ may be taken to be of the form

$$(x, y) \mapsto (x, y - q(x)),$$

where q is a real-valued Puiseux series whose leading exponent is larger than or equal to 1. Let $\tilde{f} = f \circ \varphi$ and $\tilde{g} = g \circ \varphi$ be the transformed functions in these coordinates. Then,

$$\begin{aligned} \tilde{f}(x', y') &= (x')^{\tilde{\alpha}_1} (y' + q(x'))^{\tilde{\beta}_1} \prod_{\nu \,;\, \nu \in I_f} (y' - (r_\nu(x') - q(x'))) \\ \tilde{g}(x', y') &= (x')^{\tilde{\alpha}_2} (y' + q(x'))^{\tilde{\beta}_2} \prod_{\mu \,;\, \mu \in I_g} (y' - (s_\mu(x') - q(x'))) \,. \end{aligned}$$

The combined set of nontrivial zeros of $\tilde{f}$ and $\tilde{g}$ is therefore given by

$$\{r_\nu(x') - q(x')\,;\, \nu \in I_f\} \cup \{s_\mu(x') - q(x')\,;\, \mu \in I_g\} \cup \{-q(x')\}\,.$$

Let

$$a_1(\varphi) < a_2(\varphi) < \cdots < a_l(\varphi) < a_{l+1}(\varphi) < \cdots \tag{2.3}$$

be the ordered sequence of leading exponents of roots in the above set. Now, for any $b \geq 1$,

$$\int_{B((\lambda,q(\lambda));\lambda^b)} \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx = \int_{D(\lambda)} \frac{|\tilde{g}(x',y')|^\epsilon}{|\tilde{f}(x',y')|^\delta}\, dy'\, dx',$$

where $D(\lambda)$ is the image of $B\left((\lambda, q(\lambda))\,;\lambda^b\right)$ under the transformation φ. Since

$$B\left((\lambda,0); \left(\frac{\lambda}{R}\right)^b\right) \subset D(\lambda)$$

for large R, the A_p condition implies

$$\sup_{\lambda<\lambda_0} \left(\int_{B\left((\lambda,0);\left(\frac{\lambda}{R}\right)^b\right)} \frac{|\tilde{g}(x',y')|^\epsilon}{|\tilde{f}(x',y')|^\delta}\, dy',\, dx'\right) \left(\int_{B\left((\lambda,0);\left(\frac{\lambda}{R}\right)^b\right)} \frac{|\tilde{f}(x',y')|^{\delta\frac{p'}{p}}}{|\tilde{g}(x',y')|^{\epsilon\frac{p'}{p}}}\, dy',\, dx'\right)^{\frac{p}{p'}} < \infty$$

for some fixed large constant R.

Thus the problem reduces to the one that has already been considered. Choosing b in various ranges as shown before, i.e.,

$$\begin{aligned} \max\left(1, a_l(\varphi)\right) < b < a_{l+1}(\varphi) &\quad \text{if } a_{l+1}(\varphi) > 1\,, \\ a_l(\varphi) < b < a_{l+1}(\varphi) &\quad \text{if } a_l < 1, \end{aligned}$$

one obtains condition (2) for a general admissible transformation φ.

Next, we shall prove sufficiency, i.e., if (ϵ, δ) belongs to the region specified by conditions (1) and (2) of the theorem, then we shall show that $|g|^\epsilon/|f|^\delta \in A_p(K)$. The proof depends on the classification of the balls in a neighborhood of the origin into a finite number of classes depending on the location of their center (measured by the proximity to the zero variety of f and g) and their relative sizes (measured by the roots that fall within the ball). Uniform bounds of the form (0.4) are then obtained for balls in each

class. **Step 1:**
Here we show that

$$\sup_{\lambda<\lambda_0}\left(\frac{1}{\lambda^2}\int_{B(0;\lambda)}\frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\,dy\,dx\right)\left(\frac{1}{\lambda^2}\int_{B(0;\lambda)}\frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}} < \infty$$

i.e.,

$$\sup_{\lambda<\lambda_0}\left(\int_{B(0;1)}\frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx\right)\left(\int_{B(0;1)}\frac{|f(\lambda x,\lambda y)|^{\delta\frac{p'}{p}}}{|g(\lambda x,\lambda y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}} < \infty$$

for λ_0 sufficiently small.

To ease the notation we shall henceforth assume, without loss of generality, that $\tilde{\alpha}_i = \tilde{\beta}_i = 0$ for $i = 1, 2$. Then,

$$\begin{aligned} f(\lambda x,\lambda y) &= \prod_{\nu\in I_f}(\lambda y - r_\nu(\lambda x)) \\ &= \lambda^M \prod_{\nu\,;\,a_\nu\geq 1}\left(y - \frac{1}{\lambda}r_\nu(\lambda x)\right)\prod_{\nu\,;\,a_\nu<1}\left(y\lambda^{1-a_\nu} - \frac{1}{\lambda^{a_\nu}}r_\nu(\lambda x)\right), \end{aligned}$$

where M is an integer depending on f. Similarly,

$$\begin{aligned} g(\lambda x,\lambda y) &= \prod_{\mu\in I_g}(\lambda y - s_\mu(\lambda x)) \\ &= \lambda^N \prod_{\mu\,;\,a_\mu\geq 1}\left(y - \frac{1}{\lambda}s_\mu(\lambda x)\right)\prod_{\mu\,;\,a_\mu<1}\left(y\lambda^{1-a_\mu} - \frac{1}{\lambda^{a_\mu}}s_\mu(\lambda x)\right), \end{aligned}$$

where s_μ-s are the roots of g and N is an integer depending on g. Therefore,

$$\begin{aligned} &\int_{B(0;1)}\frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx = \lambda^{N\epsilon - M\delta}\times \\ &\quad\int_{B(0;1)}\frac{|\prod_{\mu\,;a_\mu\geq 1}(y-\lambda^{-1}s_\mu(\lambda x))\prod_{\mu\,;a_\mu<1}(y\lambda^{1-a_\mu}-\lambda^{-a_\mu}s_\mu(\lambda x))|^\epsilon}{|\prod_{\nu\,;a_\nu\geq 1}(y-\lambda^{-1}r_\nu(\lambda x))\prod_{\nu\,;a_\nu<1}(y\lambda^{1-a_\nu}-\lambda^{-a_\nu}r_\nu(\lambda x))|^\delta}\,dy\,dx \end{aligned} \tag{2.4}$$

and

$$\int_{B(0;1)} \frac{|f(\lambda x,\lambda y)|^{\delta\frac{p'}{p}}}{|g(\lambda x,\lambda y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx = \lambda^{(M\delta-N\epsilon)\frac{p'}{p}}\times$$

$$\int_{B(0;1)} \frac{|\prod_{\nu;a_\nu\geq 1}(y-\lambda^{-1}r_\nu(\lambda x))\prod_{\nu;a_\nu<1}(y\lambda^{1-a_\nu}-\lambda^{-a_\nu}r_\nu(\lambda x))|^{\delta\frac{p'}{p}}}{|\prod_{\mu;a_\mu\geq 1}(y-\lambda^{-1}s_\mu(\lambda x))\prod_{\mu;a_\mu<1}(y\lambda^{1-a_\mu}-\lambda^{-a_\mu}s_\mu(\lambda x))|^{\epsilon\frac{p'}{p}}}\,dy\,dx.$$

We would be done if we show that

$$\sup_{\lambda<\lambda_0}\lambda^{M\delta-N\epsilon}\int_{B(0;1)}\frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx \;<\;\infty$$

and

$$\sup_{\lambda<\lambda_0}\lambda^{(N\epsilon-M\delta)\frac{p'}{p}}\int_{B(0;1)}\frac{|f(\lambda x,\lambda y)|^{\delta\frac{p'}{p}}}{|g(\lambda x,\lambda y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx \;<\;\infty$$

for λ_0 sufficiently small. By symmetry, it suffices to show finiteness for just one of the above integrals, let us say the first one. For this, we shall decompose the domain of integration $B(0;1)$ into several subregions depending on the relative sizes of x and y as in the proof of Theorem 1 of [Pra]. In the sequel, we use the notation $A \ll B$ to mean that $A \leq C_1 B$ for some small constant C_1 depending on f and g. $A \gg B$ implies $A \geq C_2 B$, and $C_1 B \leq A \leq C_2 B$ is denoted by $A \sim B$.
Case 1: $\lambda^{a_{l+1}-1}|x|^{a_{l+1}} \ll |y| \ll \lambda^{a_l-1}|x|^{a_l}$, $a_l \geq 1$.
Since

$$r_\nu(x) = c_\nu x^{a_\nu} + o(|x|^{a_\nu}),$$

we have

$$\frac{1}{\lambda}r_\nu(\lambda x) = c_\nu\lambda^{a_\nu-1}x^{a_\nu} + o(\lambda^{a_\nu-1}|x|^{a_\nu}) \quad \text{if} \quad a_\nu \geq 1, \text{ and}$$
$$\frac{1}{\lambda^{a_\nu}}r_\nu(\lambda x) = c_\nu x^{a_\nu} + \frac{1}{\lambda^{a_\nu}}o(|\lambda x|^{a_\nu}) \quad \text{if} \quad a_\nu < 1.$$

Let

$$R_1(\lambda,l) := \{(x,y)\in B(0;1);\lambda^{a_{l+1}-1}|x|^{a_{l+1}} \ll |y| \ll \lambda^{a_l-1}|x|^{a_l}\}.$$

Then, on $R_1(\lambda, l)$, the following estimates hold :

$$|y\lambda^{1-a_\nu} - \lambda^{-a_\nu} r_\nu(\lambda x)| \sim |x|^{a_\nu} \text{ if } a_\nu < 1$$

$$|y - \lambda^{-1} r_\nu(\lambda x)| \sim \begin{cases} \lambda^{a_\nu - 1}|x|^{a_\nu} & \text{if} \quad 1 \le a_\nu \le a_l \\ |y| & \text{if} \quad a_\nu > a_l. \end{cases}$$

Therefore,

$$\frac{1}{\lambda^{N\epsilon - M\delta}} \int_{R_1(\lambda,l)} \frac{|g(\lambda x, \lambda y)|^\epsilon}{|f(\lambda x, \lambda y)|^\delta} \, dy \, dx \tag{2.5}$$

$$\sim \lambda^{\epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) m_{l'}} \int_{R_1(\lambda,l)} |x|^{\epsilon C_l - \delta A_l} |y|^{\epsilon D_l - \delta B_l} \, dy \, dx$$

$$\sim \lambda^{\epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) m_{l'}} \int_{-1}^{1} |x|^{\epsilon C_l - \delta A_l} \left(\lambda^{a_l - 1} x^{a_l}\right)^{D_l \epsilon - B_l \delta + 1} \, dx$$

since $D_l \epsilon - B_l \delta + 1 > 0$. The integral in (2.5) is therefore of size comparable to

$$\lambda^{\epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'}-1) m_{l'} + (a_l - 1)(D_l \epsilon - B_l \delta + 1)}$$

$$\times \int_{-1}^{1} |x|^{\epsilon(C_l + a_l D_l) - \delta(A_l + a_l B_l) + a_l} \, dx.$$

We note that our assumption $(\epsilon, \delta) \in S(g, f)$ implies that the above integral in x converges. Thus, we only need to check that the exponent of λ, namely

$$F_l := \epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'} - 1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_l} (a_{l'} - 1) m_{l'} + (a_l - 1)(D_l \epsilon - B_l \delta + 1)$$

is non-negative. Now,

$$\begin{aligned} F_l &= \epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) m_{l'} \\ &\qquad + (a_l - 1) \left(\epsilon(D_l + n_l) - \delta(B_l + m_l) + 1\right) \\ &= \epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) m_{l'} \\ &\qquad + (a_l - 1) \left(\epsilon D_{l-1} - \delta B_{l-1} + 1\right) \\ &\ge \epsilon \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) n_{l'} - \delta \sum_{l' \,;\, 1 \le a_{l'} \le a_{l-1}} (a_{l'} - 1) m_{l'} \\ &\qquad + (a_{l-1} - 1)(\epsilon D_{l-1} - \delta B_{l-1} + 1) := F_{l-1} \end{aligned}$$

since $a_l > a_{l-1}$ and $\epsilon D_{l-1} - \delta B_{l-1} + 1 > 0$ by assumption. Thus, an inductive argument shows that

$$F_l \geq F_{\tilde{l}} = (a_{\tilde{l}} - 1)(\epsilon D_{\tilde{l}-1} - \delta B_{\tilde{l}-1} + 1) \geq 0.$$

where $\tilde{l} := \min\{l\,;\, a_l \geq 1\}$. The integral in (2.5) is therefore finite and bounded in λ for λ sufficiently small. This completes case 1. We note that the same arguments apply when $y \gg \lambda^{a_1-1}x^{a_1}$ and $y \ll \lambda^{a_L-1}x^{a_L}$ where $a_L := \max_l a_l$.

Case 2 : $|y|^{\frac{1}{a_l}}\lambda^{\frac{1}{a_l}-1} \ll |x| \ll |y|^{\frac{1}{a_{l+1}}}\lambda^{\frac{1}{a_{l+1}}-1},\ a_{l+1} \leq 1$

Let

$$R_2(\lambda, l) := \left\{(x,y) \in B(0;1);\, |y|^{\frac{1}{a_l}}\lambda^{\frac{1}{a_l}-1} \ll |x| \ll |y|^{\frac{1}{a_{l+1}}}\lambda^{\frac{1}{a_{l+1}}-1}\right\}.$$

On $R_2(\lambda, l)$ we have :

$$|y - \lambda^{-1}r_\nu(\lambda x)| \sim |y| \text{ if } a_\nu \geq 1$$

$$|y\lambda^{1-a_\nu} - \lambda^{-a_\nu}r_\nu(\lambda x)| \sim \begin{cases} |x|^{a_\nu} & \text{if} \quad a_\nu \leq a_l < 1 \\ \lambda^{1-a_\nu}|y| & \text{if} \quad 1 \geq a_\nu > a_l. \end{cases}$$

Therefore,

$$\frac{1}{\lambda^{N\epsilon - M\delta}} \int_{R_2(\lambda,l)} \frac{|g(\lambda x, \lambda y)|^\epsilon}{|f(\lambda x, \lambda y)|^\delta}\, dy\, dx \tag{2.6}$$

$$\sim \lambda^{\epsilon\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})n_{l'} - \delta\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})m_{l'}} \int_{R_2(\lambda,l)} |x|^{\epsilon C_l - \delta A_l}|y|^{\epsilon D_l - \delta B_l}\, dy\, dx$$

$$\sim \lambda^{\epsilon\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})n_{l'} - \delta\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})m_{l'}} \times$$
$$\int_{-1}^{1} |y|^{\epsilon D_l - \delta B_l}\left(\lambda^{\frac{1}{a_{l+1}}-1}y^{\frac{1}{a_{l+1}}}\right)^{C_l\epsilon - A_l\delta + 1} dy$$
$$(\text{since } \epsilon C_l - \delta A_l + 1 > 0)$$

$$\sim \lambda^{\epsilon\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})n_{l'} - \delta\sum_{l'\,;\,1>a_{l'}\geq a_l}(1-a_{l'})m_{l'} + \left(\frac{1}{a_{l+1}}-1\right)(\epsilon C_l - \delta A_l + 1)}$$
$$\times \int_{-1}^{1} y^{\epsilon\left(D_l + \frac{C_l}{a_{l+1}}\right) - \delta\left(B_l + \frac{A_l}{a_{l+1}}\right) + \frac{1}{a_{l+1}}}\, dy.$$

The integral in y converges because $(\epsilon, \delta) \in S(g, f)$. As before, we use condition (2) to determine the sign of the exponent of λ:

$$\begin{aligned}
\tilde{F}_l &:= \epsilon \sum_{l'\,;\,1>a_{l'}\geq a_l} (1-a_{l'})\, n_{l'} - \delta \sum_{l'\,;\,1>a_{l'}\geq a_l} (1-a_{l'})\, m_{l'} \\
&\qquad\qquad\qquad + \left(\frac{1}{a_{l+1}} - 1\right)(\epsilon C_l - \delta A_l + 1) \\
&= \epsilon \sum_{l'\,;\,1>a_{l'}\geq a_{l+1}} (1-a_{l'})\, n_{l'} - \delta \sum_{l'\,;\,1>a_{l'}\geq a_{l+1}} (1-a_{l'})\, m_{l'} \\
&\qquad\qquad\qquad + \left(\frac{1}{a_{l+1}} - 1\right)(\epsilon C_{l+1} - \delta A_{l+1} + 1) \\
&\geq \epsilon \sum_{l'\,;\,1>a_{l'}\geq a_{l+1}} (1-a_{l'})\, n_{l'} - \delta \sum_{l'\,;\,1>a_{l'}\geq a_{l+1}} (1-a_{l'})\, m_{l'} \\
&\qquad\qquad\qquad + \left(\frac{1}{a_{l+2}} - 1\right)(\epsilon C_{l+1} - \delta A_{l+1} + 1) \\
&= \tilde{F}_{l+1} \geq \cdots \\
&\geq \tilde{F}_{l^*} = \epsilon(1-a_{l^*})n_{l^*} - \delta(1-a_{l^*})m_{l^*} + \left(\frac{1}{a_{l^*}} - 1\right)(\epsilon C_{l^*-1} - \delta A_{l^*-1} + 1) \\
&\qquad\qquad \text{where } l^* = \max\{l\,;\, a_l < 1\} \\
&= \left(\frac{1}{a_{l^*}} - 1\right)(\epsilon C_{l^*} - \delta A_{l^*} + 1)
\end{aligned}$$

which is positive. This proves the boundedness of the integral in (2.6) for small λ.
Case 3 :$\lambda^{a_{l+1}-1}|x|^{a_{l+1}} \ll |y| \ll \lambda^{a_l-1}|x|^{a_l}$, $a_l < 1$, $a_{l+1} \geq 1$.
As before, we define

$$R_3(\lambda, l) := \left\{(x,y) \in B(0;1)\,;\, \lambda^{a_{l+1}-1}|x|^{a_{l+1}} \ll |y| \ll \lambda^{a_l-1}|x|^{a_l}\right\}.$$

Then,

$$\frac{1}{\lambda^{\epsilon N-\delta M}}\int_{R_3(\lambda,l)}\frac{|g(\lambda x,\lambda y)|^{\epsilon}}{|f(\lambda x,\lambda y)|^{\delta}}\,dy\,dx$$
$$\sim\int_{R_3(\lambda,l)}|x|^{\epsilon C_l-\delta A_l}|y|^{\epsilon D_l-\delta B_l}\,dy\,dx$$
$$\sim\left\{\begin{array}{ll}\displaystyle\int_{-1}^{1}|x|^{\epsilon C_l-\delta A_l}\max\left[\left(\lambda^{a_l-1}|x|^{a_l}\right)^{\epsilon D_l-\delta B_l+1},\left(\lambda^{a_{l+1}-1}|x|^{a_{l+1}}\right)^{\epsilon D_l-\delta B_l+1}\right]\,dx & \text{if } \lambda^{a_l-1}|x|^{a_l}\lesssim 1,\\ \displaystyle\int_{-1}^{1}|x|^{\epsilon C_l-\delta A_l}\max\left[1,\left(\lambda^{a_{l+1}-1}|x|^{a_{l+1}}\right)^{\epsilon D_l-\delta B_l+1}\right]\,dx & \text{if } \lambda^{a_l-1}|x|^{a_l}\gtrsim 1.\end{array}\right\}$$

Let us first consider the case when $\lambda^{a_l-1}|x|^{a_l}\lesssim 1$. Then the integral above is of size comparable to

$$\max\Big[\lambda^{\left(\frac{1}{a_l}-1\right)(\epsilon C_l-\delta A_l+1)},\lambda^{(a_{l+1}-1)(\epsilon D_l-\delta B_l+1)}\times\lambda^{\left(\frac{1}{a_l}-1\right)[\epsilon(C_{l+1}+a_{l+1}D_{l+1})-\delta(A_{l+1}+a_{l+1}B_{l+1})+a_{l+1}+1]}\Big]. \quad (2.7)$$

We know, by assumption, that

$$\epsilon C_l-\delta A_l+1>0 \qquad \text{for } a_l<1$$

while

$$\begin{aligned}(a_{l+1}-1)\,(\epsilon D_l-\delta B_l+1)&=0 \quad \text{if } a_{l+1}=1\\ &>0 \quad \text{if } a_{l+1}>1.\end{aligned}$$

Also,

$$\epsilon\,(C_{l+1}+a_{l+1}D_{l+1})-\delta\,(A_{l+1}+a_{l+1}B_{l+1})+a_{l+1}+1>0$$

since $(\epsilon,\delta)\in S(g,f)$. The expression in (2.7) is therefore bounded.

Next, let us consider the situation when $\lambda^{a_l-1}|x|^{a_l}\gtrsim 1$. A similar computation as above shows that the integral in this case is of size comparable to

$$\max\left[\lambda^{\left(\frac{1}{a_l}-1\right)(\epsilon C_l-\delta A_l+1)},\lambda^{(a_{l+1}-1)(\epsilon D_l-\delta B_l+1)}\right]$$

Once again, our assumption implies that the integral is bounded above for small λ.

Case 4 : $|y| \sim \lambda^{a_l-1}|x|^{a_l}, a_l \geq 1$
Let
$$R_4(\lambda,l) := \{(x,y) \in B(0;1)\,;\, |y| \sim \lambda^{a_l-1}|x|^{a_l}\}.$$
Using the algorithm described in the proof of Theorem 1 of [Pra] we decompose the region $R_4(\lambda,l)$ as
$$R_4(\lambda,l) = \bigcup_{i\in\Gamma(l)} R_4^i(\lambda,l),$$
where $\Gamma(l)$ is a finite index set ranging over the different steps involved in the process of resolution of the cluster $\mathcal{S}_l$. For every $i \in \Gamma(l)$ there exists an admissible transformation φ_i associated to the weighted integral in (2.4) and a positive integer l_i satisfying the following properties :
$$\varphi_i \,:\, (x,y) \mapsto \left(x, y - \frac{1}{\lambda}q_i(\lambda x)\right) \quad \text{for some } q_i \in \mathcal{S}_l,$$
and
$$R_4^i(\lambda,l) := \left\{(x,y) \in B(0;1)\,;\, \lambda^{a_{l_i+1}(\varphi_i)-1}|x|^{a_{l_i+1}(\varphi_i)} \ll |y - \frac{1}{\lambda}q_i(\lambda x)| \ll \lambda^{a_{l_i}(\varphi_i)-1}|x|^{a_{l_i}(\varphi_i)}\right\}.$$
Here, the elements of the sequence
$$a_1(\varphi_i) < a_2(\varphi_i) < \cdots < a_{l'}(\varphi_i) < a_{l'+1}(\varphi_i) < \cdots$$
are defined as in (2.3) with q replaced by q_i. It is a consequence of the decomposition algorithm that for every $i \in \Gamma(l)$,
$$\left\{\begin{array}{c} a_{l'}(\varphi_i) = a_{l'},\ m_{l'}(\varphi_i) = m_{l'},\ n_{l'}(\varphi_i) = n_{l'} \text{ if } l' \leq l-1, \\ D_{l-1}(\varphi_i) = D_{l-1},\ B_{l-1}(\varphi_i) = B_{l-1}, \\ a_{l'}(\varphi_i) \geq a_l \text{ if } l' \geq l, \\ l_i \geq l. \end{array}\right\} \tag{2.8}$$
Now,
$$\frac{1}{\lambda^{\epsilon N-\delta M}}\int_{R_4(\lambda,l)} \frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx = \frac{1}{\lambda^{\epsilon N-\delta M}}\sum_i \int_{R_4^i(\lambda,l)} \frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx$$
$$= \sum_i \int_{R_4^i(\lambda,l)} \frac{|\prod_{\mu\in I_g}(y-\lambda^{-1}s_\mu(\lambda x))|^\epsilon}{|\prod_{\nu\in I_f}(y-\lambda^{-1}r_\nu(\lambda x))|^\delta}\,dy\,dx.$$

Applying the change of variables

$$\varphi_i\,;\,x\mapsto x,\quad y\mapsto y_i=y-\frac{1}{\lambda}q_i(\lambda x)$$

in the i-th term reduces the above sum to

$$\sum_i\int_{\varphi_i\left(R_4^i(\lambda,l)\right)}\frac{|\prod_{\mu\in I_g}\left[y_i-\lambda^{-1}\left(s_\mu\left(\lambda x\right)-q_i\left(\lambda x\right)\right)\right]|^\epsilon}{|\prod_{\nu\in I_f}\left[y_i-\lambda^{-1}\left(r_\nu\left(\lambda x\right)-q_i\left(\lambda x\right)\right)\right]|^\delta}\,dy_i\,dx$$
$$\sim\sum_i\lambda^{F_{l_i}(\varphi_i)}\int_{-1}^{1}|x|^{\epsilon\left[C_{l_i}(\varphi_i)+a_{l_i}(\varphi_i)D_{l_i}(\varphi_i)\right]-\delta\left[A_{l_i}(\varphi_i)+a_{l_i}(\varphi_i)B_{l_i}(\varphi_i)\right]+a_{l_i}(\varphi_i)}dx,$$

where $a_l(\varphi_i)$, $A_l(\varphi_i)$, $B_l(\varphi_i)$, $C_l(\varphi_i)$, $D_l(\varphi_i)$, $F_l(\varphi_i)$ are the quantities $a_l, A_l, B_l, C_l, D_l, F_l$ respectively, computed in the coordinates φ_i. The expression in the last step above follows from estimates similar to the ones obtained in case 1. The integral in x converges because $(\epsilon,\delta)\in S(g,f)$, while a repetition of the monotonicity argument presented in case 1 shows that

$$F_{l_i}\left(\varphi_i\right)\geq F_{l_i-1}\left(\varphi_i\right)\geq\cdots\geq F_l\left(\varphi_i\right).$$

But (2.8) implies that

$$F_l(\varphi_i)=F_l$$

for every $i\in\Gamma(l)$, and we have seen in case 1 that F_l is non-negative. Thus one concludes that

$$\sup_{\lambda<\lambda_0}\frac{1}{\lambda^{\epsilon N-\delta M}}\int_{R_4(\lambda,l)}\frac{|g(\lambda x,\lambda y)|^\epsilon}{|f(\lambda x,\lambda y)|^\delta}\,dy\,dx<\infty$$

for λ_0 sufficiently small.

Case 5 : $y\sim\lambda^{a_l-1}|x|^{a_l}, a_l<1$

We omit the details of this case, as they are remarkably similar to case 4. In fact, one way of treating this case is to interchange the roles of x and y, so that a_l gets replaced by $\frac{1}{a_l}$ which is larger than 1, and then apply case 4. This completes the proof of step 1.

Our next task is to show that the expression

$$\left(\frac{1}{\lambda^2}\int_B\frac{|g|^\epsilon}{|f|^\delta}\,dy\,dx\right)\left(\frac{1}{\lambda^2}\int_B\frac{|f|^{\delta\frac{p'}{p}}}{|g|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}}$$

is bounded for all balls $B = B((x_0, y_0); \lambda) \subset K$ such that $(x_0, y_0) \neq 0$. If $|x_0| + |y_0| \lesssim \lambda$, then $B \subseteq B(0; C\lambda)$ for a suitable constant C and we may invoke step 1 to obtain the desired conclusion. Thus, it suffices to consider balls B for which $|x_0| + |y_0| \gg \lambda$. Without loss of generality, we may assume that $|y_0| \lesssim |x_0|$. Balls satisfying this property may be further subdivided according to the following restrictions :

- $|y_0| \ll |x_0|^{a_N}$, $|x_0|^{a_{l+1}} \ll |y_0| \ll |x_0|^{a_l}$ for $l \geq 1$, or $|y_0| \gg |x_0|^{a_1}$, or
- $|y_0| \sim |x_0|^{a_l}$, $l \geq 1$.

Step 2 :
Here we show that

$$\sup_{\substack{\lambda<\lambda_0 \\ (x_0,y_0)\,;\,|x_0|^{a_{l+1}}\ll|y_0|\ll|x_0|^{a_l}}} \left(\frac{1}{\lambda^2}\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\,dy\,dx\right)\left(\frac{1}{\lambda^2}\int_B \frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}} < \infty \tag{2.9}$$

for all l with $a_l \geq 1$.
Case 1 : $\lambda \lesssim |x_0|^{a_{l+1}}$
In this case,

$$B \subseteq \{(x,y) \in K\,;\, |x|^{a_{l+1}} \ll |y| \ll |x|^{a_l}\}$$

and hence,

$$|f(x,y)| \sim |x|^{A_l}|y|^{B_l} \sim |x_0|^{A_l}|y_0|^{B_l}$$
$$g(x,y) \sim |x|^{C_l}|y|^{D_l} \sim |x_0|^{C_l}|y_0|^{D_l}.$$

Clearly (2.9) holds in this case.
Case 2 : $|x_0|^{a_{l+1}} \ll \lambda \ll |x_0|^{a_l}$
Here we have

$$B \subseteq \{(x,y) \in K\,;\, |y| \ll |x|^{a_l}\},$$

which roughly means that the roots of f and g that "cross" the ball B cannot have leading exponent less than or equal to a_l. Therefore,

$$\frac{1}{\lambda^2}\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta} \lesssim \frac{1}{\lambda^2}\int_{x_0-\lambda}^{x_0+\lambda}\int_{y_0-\lambda}^{y_0+\lambda} \frac{|\prod_{\mu\,;\,a_\mu\geq a_{l+1}}(y-s_\mu(x))|^\epsilon}{|\prod_{\nu\,;\,a_\nu\geq a_{l+1}}(y-r_\nu(x))|^\delta}\,dy\,|x|^{\epsilon C_l-\delta A_l}\,dx$$
$$=\frac{1}{\lambda^2}\left(\sum_{l'\,;\,l'\geq l}\mathcal{I}_{l'} + \sum_{l'\,;\,l'\geq l+1}\mathcal{J}_{l'}\right)$$

where

$$\mathcal{I}_{l'} := \int_{[x_0-\lambda,\, x_0+\lambda]\times[y_0-\lambda,\, y_0+\lambda]\cap R_1(1,l')} \frac{|\prod_{\mu\,;\,a_\mu\geq a_{l+1}}(y-s_\mu(x))\,|^\epsilon}{|\prod_{\nu\,;\,a_\nu\geq a_{l+1}}(y-r_\nu(x))\,|^\delta}\;|x|^{\epsilon C_l-\delta A_l}\,dy\,dx,$$

and

$$\mathcal{J}_{l'} := \int_{[x_0-\lambda,\, x_0+\lambda]\times[y_0-\lambda,\, y_0+\lambda]\cap R_4(1,l')} \frac{|\prod_{\mu\,;\,a_\mu\geq a_{l+1}}(y-s_\mu(x))\,|^\epsilon}{|\prod_{\nu\,;\,a_\nu\geq a_{l+1}}(y-r_\nu(x))\,|^\delta}\;|x|^{\epsilon C_l-\delta A_l}\,dy\,dx.$$

Using estimates for the integrand obtained in step 1, we obtain the following size estimates for $\mathcal{I}_{l'}$:

$$\mathcal{I}_{l'} \sim |x_0|^{\epsilon(C_{l'}+a_{l'}D_{l'})-\delta(A_{l'}+a_{l'}B_{l'})+a_l}\lambda \qquad \text{if } l'\geq l+1,$$

while

$$\begin{aligned}\mathcal{I}_l &\sim \int_{x_0-\lambda}^{x_0+\lambda}|x|^{\epsilon C_l-\delta A_l}\int_{[y_0-\lambda,y_0+\lambda]\cap\{y\,;\,(x,y)\in R_1(1,l)\}}|y|^{\epsilon D_l-\delta B_l}\,dy\\ &\sim \left\{\begin{array}{ll}\displaystyle\int_{x_0-\lambda}^{x_0+\lambda}|x|^{\epsilon C_l-\delta A_l}\lambda^{\epsilon D_l-\delta B_l+1}\,dx & \text{if } \lambda\gtrsim|y_0|\\ \displaystyle\int_{x_0-\lambda}^{x_0+\lambda}|x|^{\epsilon C_l-\delta A_l}|y_0|^{\epsilon D_l-\delta B_l}\lambda\,dx & \text{if } \lambda\lesssim|y_0|\end{array}\right\}\\ &\sim \left\{\begin{array}{ll}|x_0|^{\epsilon C_l-\delta A_l}\lambda^{\epsilon D_l-\delta B_l+2} & \text{if } \lambda\gtrsim|y_0|\\ |x_0|^{\epsilon C_l-\delta A_l}y_0^{\epsilon D_l-\delta B_l}\lambda^2 & \text{if } \lambda\lesssim|y_0|\end{array}\right\}.\end{aligned}$$

On the other hand, the resolution scheme in the proof of Theorem 1 of [Pra] provides a decomposition of $R_4(1,l')$, as we have observed in case 4 of step 1. Therefore $\mathcal{J}_{l'}$ may be bounded by a sum of integrals as follows :

$$\mathcal{J}_{l'} \leq \sum_{i\in\Gamma(l')}\mathcal{J}_{l'}^i \qquad \text{for every } l'\geq l+1,$$

where

$$\mathcal{J}_{l'}^i := \int_{[x_0-\lambda,\, x_0+\lambda]\times[y_0-\lambda,\, y_0+\lambda]\cap R_4^i(1,l')} \frac{|\prod_{\mu\,;\,a_\mu\geq a_{l+1}}(y-s_\mu(x))\,|^\epsilon}{|\prod_{\nu\,;\,a_\nu\geq a_{l+1}}(y-r_\nu(x))\,|^\delta}\;|x|^{\epsilon C_l-\delta A_l}\,dy\,dx.$$

It is not difficult to see, using arguments employed in case 1 of step 1, that $\mathcal{J}_{l'}^i$ is easily estimable for every $i\in\Gamma(l')$. In fact, one obtains after the standard change of variables φ_i that

$$\mathcal{J}_{l'}^i \sim \lambda|x_0|^{\epsilon\left[C_{l_i}(\varphi_i)+a_{l_i}(\varphi_i)D_{l_i}(\varphi_i)\right]-\delta\left[A_{l_i}(\varphi_i)+a_{l_i}(\varphi_i)B_{l_i}(\varphi_i)\right]+a_{l_i}(\varphi_i)}.$$

We have therefore bounded the first integral in (2.9) by a sum of terms whose individual sizes are under control. The problem is now to determine the dominant term. We claim that

$$\left(\sum_{l'\,;\,l'\geq l} \mathcal{I}_{l'} + \sum_{l'\,;\,l'\geq l+1} \mathcal{J}_{l'}\right) \sim \mathcal{I}_l.$$

To prove our claim, we shall first show that

$$\mathcal{I}_{l'} \lesssim \mathcal{I}_l \qquad \text{for} \quad l' > l.$$

For this, we note that for every l' with $a_{l'} \geq 1$,

$$\begin{aligned}
&[\epsilon(C_{l'+1} + a_{l'+1}D_{l'+1}) - \delta(A_{l'+1} + a_{l'+1}B_{l'+1}) + a_{l'+1}] \\
&\qquad\qquad\qquad\qquad - [\epsilon(C_{l'} + a_{l'}D_{l'}) - \delta(A_{l'} + a_{l'}B_{l'}) + a_{l'}] \\
&= (a_{l'+1} - a_{l'})(\epsilon D_{l'} - \delta B_{l'} + 1) \\
&> 0.
\end{aligned}$$

Therefore, for $l' > l$,

$$\begin{aligned}
\mathcal{I}_{l'} &\lesssim \mathcal{I}_{l+1} \\
&\sim \lambda |x_0|^{[\epsilon(C_{l+1} + a_{l+1}D_{l+1}) - \delta(A_{l+1} + a_{l+1}B_{l+1}) + a_{l+1}]} \\
&\lesssim |x_0|^{\epsilon C_l - \delta A_l} \lambda^2 \min\left(\lambda^{\epsilon D_l - \delta B_l}, |y_0|^{\epsilon D_l - \delta B_l}\right),
\end{aligned}$$

where the last inequality follows from

$$\epsilon D_l - \delta B_l + 1 > 0 \quad \text{and} \quad \lambda, |y_0| \gg |x_0|^{a_{l+1}}.$$

In other words, $\mathcal{I}_{l'} \lesssim \mathcal{I}_l, \quad l' > l.$

The proof for showing that

$$\mathcal{J}^i_{l'} \lesssim \mathcal{I}_l \quad \text{ for all } l' \geq l+1 \text{ and all } i \in \Gamma(l')$$

is similar. Condition (2) implies that for every $i \in \Gamma(l')$,

$$\epsilon \left[C_r(\varphi_i) + a_r(\varphi_i)D_r(\varphi_i)\right] - \delta \left[A_r(\varphi_i) + a_r(\varphi_i)B_r(\varphi_i)\right] + a_r(\varphi_i)$$

is an increasing function of r in the range where $a_r(\varphi_i) \geq 1$. Since $l_i \geq l'$ for every $i \in \Gamma(l')$, we have

$$\begin{aligned}
&\epsilon\left[C_{l_i}(\varphi_i) + a_{l_i}(\varphi_i)\, D_{l_i}(\varphi_i)\right] - \delta\left[A_{l_i}(\varphi_i) + a_{l_i}(\varphi_i)\, B_{l_i}(\varphi_i)\right] + a_{l_i}(\varphi_i) \\
\geq\, &\epsilon\left[C_{l'}(\varphi_i) + a_{l'}(\varphi_i)\, D_{l'}(\varphi_i)\right] - \delta\left[A_{l'}(\varphi_i) + a_{l'}(\varphi_i)\, B_{l'}(\varphi_i)\right] + a_{l'}(\varphi_i) \\
=\, &\epsilon\left[C_{l'-1}(\varphi_i) + a_{l'}(\varphi_i)\, D_{l'-1}(\varphi_i)\right] - \delta\left[A_{l'-1}(\varphi_i) + a_{l'}(\varphi_i)\, B_{l'-1}(\varphi_i)\right] + a_{l'}(\varphi_i) \\
=\, &\epsilon C_{l'-1} - \delta A_{l'-1} + a_{l'}(\varphi_i)\left[\epsilon D_{l'-1} - \delta B_{l'-1} + 1\right] \\
\geq\, &\epsilon\left(C_{l'-1} + a_{l'} D_{l'-1}\right) - \delta\left(A_{l'-1} + a_{l'} B_{l'-1}\right) + a_{l'} \\
=\, &\epsilon\left(C_{l'} + a_{l'} D_{l'}\right) - \delta\left(A_{l'} + a_{l'} B_{l'}\right) + a_{l'}.
\end{aligned}$$

In the above computations, we have made use of condition (2) and properties of the decomposition procedure listed in (2.8). Thus, for $l' \geq l+1$,

$$\mathcal{J}_{l'} \lesssim \mathcal{I}_{l'} \lesssim \mathcal{I}_l.$$

In conclusion, we have,

$$\begin{aligned}
\frac{1}{\lambda^2}\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx &\sim \frac{1}{\lambda^2}\mathcal{I}_l \\
&\sim \left\{\begin{array}{ll} |x_0|^{\epsilon C_l - \delta A_l}\lambda^{\epsilon D_l - \delta B_l} & \text{if } \lambda \gtrsim |y_0| \\ |x_0|^{\epsilon C_l - \delta A_l}|y_0|^{\epsilon D_l - \delta B_l} & \text{if } \lambda \lesssim |y_0| \end{array}\right\}
\end{aligned}$$

while

$$\frac{1}{\lambda^2}\int_B \frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\, dy\, dx \sim \left\{\begin{array}{ll} |x_0|^{(\delta A_l - \epsilon C_l)\frac{p'}{p}}\lambda^{(\delta B_l - \epsilon D_l)\frac{p'}{p}} & \text{if } \lambda \gtrsim |y_0| \\ |x_0|^{(\delta A_l - \epsilon C_l)\frac{p'}{p}}|y_0|^{(\delta B_l - \epsilon D_l)\frac{p'}{p}} & \text{if } \lambda \lesssim |y_0| \end{array}\right\}$$

This completes the proof of case 2.

Case 3 : $\lambda \sim |x_0|^{a_l}$

In this case, the roots crossing the ball B may have leading exponents greater than or equal to a_l, so arguments similar to case 2 show that

$$\frac{1}{\lambda^2}\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx \;\sim\; |x_0|^{\epsilon(C_l + a_l D_l) - \delta(A_l + a_l B_l)}$$

The cases $|x_0|^{a_{l'+1}} \ll \lambda \ll |x_0|^{a_{l'}}$, $\lambda \sim |x_0|^{a_{l'}}$ for $l' \leq l$ are all treated by repeated applications of the arguments given above. The proof of step 2 is therefore complete.

Step 3 :

In this final step, we show that

$$\sup_{\substack{\lambda<\lambda_0 \\ (x_0,y_0)\,;\,|y_0|\sim|x_0|^{a_l}}} \left(\frac{1}{\lambda^2}\int_B \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}\,dy\,dx\right)\left(\frac{1}{\lambda^2}\int_B \frac{|f(x,y)|^{\delta\frac{p'}{p}}}{|g(x,y)|^{\epsilon\frac{p'}{p}}}\,dy\,dx\right)^{\frac{p}{p'}} < \infty \tag{2.10}$$

for all l with $a_l \geq 1$.

This step does not involve any idea beyond what has been already presented earlier in the proof, so we shall only discuss the key point that highlights the similarity with the previous cases. The idea is to use the decomposition

$$R_4(1,l) = \bigcup_{i\in\Gamma(l)} R_4^i(1,l)$$

to bound the supremum in (2.10) by the largest of several suprema, each over a domain of the form

$$(x_0,y_0) \in R_4^i(1,l),\ \lambda < \lambda_0.$$

We claim that the problem is then reduced to one of the type handled in step 2.

Fix $i \in \Gamma(l)$ and put $q_i = q$. Then,

$$\frac{1}{\lambda^2}\int_{[x_0-\lambda,x_0+\lambda]\times[y_0-\lambda,y_0+\lambda]} \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}\,dy\,dx = \frac{1}{\lambda^2}\int_{\substack{x_0-\lambda\leq x'\leq x_0+\lambda \\ y_0-\lambda-q(x')\leq y'\leq y_0+\lambda-q(x')}} \frac{|\tilde{g}(x',y')|^{\epsilon}}{|\tilde{f}(x',y')|^{\delta}}\,dy'\,dx'$$

where

$$x' = x, \quad y' = y - q(x)$$

and

$$\tilde{f}(x',y') = f(x',y'+q(x')), \quad \tilde{g}(x',y') = g(x',y'+q(x')).$$

Now, if $x_0 - \lambda \leq x' \leq x_0 + \lambda$, then by the mean-value theorem

$$|q(x) - q(x_0)| \lesssim \lambda x_0^{a_l-1} \lesssim \lambda.$$

Therefore,

$$
\begin{aligned}
&\{(x',y')\,;\, x_0-\lambda \le x' \le x_0+\lambda, y_0-\lambda-q(x') \le y' \le y_0+\lambda-q(x')\}\\
=&\big\{(x',y')\,;\, x_0-\lambda \le x' \le x_0+\lambda,\\
&\qquad -\lambda-(q(x')-q(x_0)) \le y'-(y_0-q(x_0)) \le \lambda-(q(x')-q(x_0))\big\}\\
\subseteq&\{(x',y')\,;\, x_0-R\lambda \le x' \le x_0+R\lambda, -R\lambda \le y'-y_0' \le R\lambda\}
\end{aligned}
$$

for a suitable large constant R, where $y_0' = y_0 - q(x_0)$. This implies

$$
\begin{aligned}
&\frac{1}{\lambda^2}\int_{[x_0-\lambda,x_0+\lambda]\times[y_0-\lambda,y_0+\lambda]} \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\,dy\,dx \lesssim\\
&\qquad\qquad \frac{1}{\lambda^2}\int_{[x_0-R\lambda,x_0+R\lambda]\times\left[y_0'-R\lambda,y_0'+R\lambda\right]} \frac{|\tilde g(x',y')|^\epsilon}{|\tilde f(x',y')|^\delta}\,dy'\,dx'.
\end{aligned}
$$

Since

$$|x_0|^{a_{l_i+1}(\varphi_i)} \ll |y_0'| \ll |x_0|^{a_{l_i}(\varphi_i)} \quad \text{on } R_4^i(1,l),$$

the integral on the right hand side is of the type encountered in step 2. This completes the proof of the theorem.

□

3 Remarks and Examples

It is of interest to contrast the statement of Theorem 2 with its analogue in dimension 1. For $i = 1, 2$, let h_i be a real-analytic function in a neighborhood of the origin in $\mathbb{R}$, with

$$h_i(x) = x^{K_i}\tilde h_i(x), \quad \tilde h_i(0) \ne 0.$$

We have

$$\left(\frac{1}{2r}\int_{-r}^{r} \frac{|h_2(x)|^\epsilon}{|h_1(x)|^\delta}\,dx\right)\left(\frac{1}{2r}\int_{-r}^{r} \frac{|h_1(x)|^{\delta\frac{p'}{p}}}{|h_2(x)|^{\delta\frac{p'}{p}}}\,dx\right)^{\frac{p}{p'}} < \infty \tag{3.1}$$

for a *fixed* small r if and only if

$$-1 < \epsilon K_2 - \delta K_1 < p - 1. \tag{3.2}$$

One sees from elementary considerations that (3.2) is also a necessary and sufficient condition for $|h_2|^\epsilon/|h_1|^\delta$ to be an A_p weight in $[-r,r]$. One may view this property as a certain form of "stability", where the finiteness condition (3.1) for a fixed small ball implies boundedness of the same expression for all smaller sub-balls, i.e.,

$$\sup_{I\subseteq[-r,r]}\left(\frac{1}{|I|}\int_I \frac{|h_2(x)|^\epsilon}{|h_1(x)|^\delta}\,dx\right)\left(\frac{1}{|I|}\int_I \frac{|h_1(x)|^{\delta\frac{p'}{p}}}{|h_2(x)|^{\epsilon\frac{p'}{p}}}\,dx\right)^{\frac{p}{p'}}<\infty.$$

Our discussion shows that the weighted integral in $\mathbb{R}^1$ is stable in this sense. This is however not the case in dimension 2. In other words, condition (1) is not enough to ensure that $|g|^\epsilon/|f|^\delta\in A_p(K)$ for some compact neighborhood K of the origin, as is illustrated by the following example.

Consider

$$f(x,y)=(y-x)(y-x^2)\cdots(y-x^N),\quad g(x,y)=y-x,\quad \epsilon=1.$$

where N is a large positive integer. We shall see that for this example, condition (1) in the statement of the theorem is equivalent to

$$\delta<\delta_0^{\mathcal{C}_0}(g,f,1)=\min\left(1,\min_l\frac{2(l+2)}{l(2N-l+1)}\right), \tag{3.3}$$

while condition (2) implies

$$\delta<\frac{1}{N-1}. \tag{3.4}$$

For N sufficiently large and $1\le l\le N$, one has

$$\frac{1}{N-1}<\frac{2(l+2)}{l(2N-l+1)},$$

which shows that condition (2) is, in general, more restrictive than (1).

In order to prove the inequalities (3.3) and (3.4), one needs to observe that the Newton diagram of f has vertices at the points

$$\left(\frac{l(l+1)}{2},N-l\right),\quad 0\le l\le N,$$

while the Newton diagram of g has two vertices, (0,1) and (1,0). A direct computation shows that

$$\delta_l^{-1} = \frac{l(2N-l+1)}{2(l+1)} \quad \text{and} \quad \tilde{\delta}_l^{-1} = \frac{1}{l+1}, \quad 1 \leq l \leq N.$$

Therefore, we have

$$\delta_l \left(1 + \frac{1}{\tilde{\delta}_l}\right) = \frac{2(l+2)}{l(2N-l+1)}, \quad 1 \leq l \leq N. \tag{3.5}$$

The nontrivial admissible transformations associated to the pair (f, g) are given by

$$\varphi_l : (x, y) \mapsto (x, y - x^l), \quad 1 \leq l \leq N,$$

from which the generalized Newton diagrams of $f \circ \varphi_l$ and $g \circ \varphi_l$ may also be described explicitly. For $l \geq 2$, a direct analysis of these Newton diagrams shows that the only value of $\delta_k(\varphi_l)\left(1 + (\tilde{\delta}_k(\varphi_l))^{-1}\right)$ not already listed in (3.5) is 1. For the case $l = 1$, the only new addition is 2. Putting these facts together, we get (3.3). On the other hand, under the identity transformation, $B_1 = N - 1$ and $D_1 = 0$, from which (3.4) follows.

The above example reflects the inherent "instability" of the weighted integral in $\mathbb{R}^2$, a feature we have seen is absent in $\mathbb{R}^1$. The reason for this is most apparent in Step 1 of the proof, where we employ a scaling argument to reduce the integral

$$\frac{1}{\lambda^2} \int_{B(0;\lambda)} \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta} \, dy \, dx$$

to the expression on the right hand side of (2.4) . Notice that as $\lambda \to 0$, the scaling process "clumps" together all roots of f and g into three well-separated clusters. Roots which have leading exponent strictly larger than 1 get clumped into a power of y, while roots with leading exponent strictly less than 1 merge into a power of x. Roots which have leading exponent equal to 1 converge to a line of the form $y = cx$, where c is the leading coefficient of the "parent" root. The analysis in step 1 of the proof of sufficiency shows that for $(\epsilon, \delta) \in S(g, f)$, the expression

$$\lambda^{\delta M - \epsilon N} \int_{B(0;1)} \frac{|g(\lambda x, \lambda y)|^\epsilon}{|f(\lambda x, \lambda y)|^\delta} \, dy \, dx$$

is finite for every λ, but the value of the above integral may well blow up (in λ) as roots coalesce together. The same phenomenon of clumping takes place even if we consider a combination of translation and dilation of the form

$$(x, y) \mapsto (\lambda + x\lambda^b, y\lambda^b), \tag{3.6}$$

as seen in the proof of necessity, except that the "clumping pattern" is now different and depends on the value of b relative to the leading exponents. Condition (2) enforces that the limiting value of the integrand, after suitable translations and scalings, continues to be integrable - but this may, in general, pose a restriction on the set of ϵ and δ defined in (1), as we have observed in the example.

It is also possible to view the phenomenon of instability from a different perspective. Instead of comparing the average of the weighted integral over a fixed small ball with its average over all smaller sub-balls, we may look at the integral

$$\frac{1}{\lambda^2} \int_{B((x_0,y_0);\lambda)} \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta} \, dy \, dx,$$

after suitable scaling, as a "perturbation" of another weighted integral. The "unperturbed" integral depends on the relative sizes of x_0, y_0 and λ. To fix ideas, let us consider a translation-dilation combination of the form (3.6), where b is chosen as follows :

$$\max(1, a_l) < b < a_{l+1} \text{ for some } l \text{ with } a_{l+1} > 1.$$

Then, we have seen in the proof of necessity that the function

$$\frac{f(\lambda + x\lambda^b, y\lambda^b)}{\lambda^{A_l + bB_l}}$$

may be thought of, for small λ, as an analytic perturbation of the monomial y^{B_l}. A similar interpretation holds for the expression

$$\frac{g(\lambda + x\lambda^b, y\lambda^b)}{\lambda^{C_l + bD_l}}.$$

The integral

$$\int_{B(0;1)} H_1^\lambda(x, y) \, dy \, dx, \tag{3.7}$$

defined in (2.2) may therefore be visualized as a perturbation of the weighted integral

$$\int_{B(0;1)} \frac{|y|^{\epsilon D_l}}{|y|^{\delta B_l}}\, dy\, dx. \tag{3.8}$$

One may ask whether the condition

$$\epsilon D_l - \delta B_l + 1 > 0,$$

which ensures convergence of the "unperturbed" integral (3.8) would ensure boundedness of the "perturbed" integral (3.7) (in λ) as well. The answer to that question is no, as the following example shows.

We consider

$$f(x,y) = y^N \quad \text{and} \quad g(x,y) = (y-x^2)(y-x^3)\cdots(y-x^{N+1}),$$

where N is a positive integer. Let us choose $b = 1$. Then

$$\lim_{\lambda\to 0} \frac{f(\lambda + x\lambda, y\lambda)}{\lambda^N} = \lim_{\lambda\to 0} \frac{g(\lambda + x\lambda, y\lambda)}{\lambda^N} = y^N,$$

so the necessary and sufficient condition for the "limiting" integral

$$\int_{B(0;1)} |y|^{\epsilon N - \delta N}\, dy\, dx$$

to be finite is given by

$$\delta < \epsilon + \frac{1}{N}.$$

However, the above condition is not enough to guarantee that

$$\sup_{\lambda<\lambda_0} \lambda^{\epsilon N - \delta N} \int_{B(0;1)} \frac{|g(\lambda + x\lambda, y\lambda)|^{\epsilon}}{|f(\lambda + x\lambda, y\lambda)|^{\delta}}\, dy\, dx \ < \ \infty.$$

Indeed, the integral above converges for a fixed λ only if

$$\delta < \frac{1}{N}.$$

Notice that the clumping phenomenon is again at play here, but in reverse. The scaling scenario gives rise to perturbation of a certain form, which leaves the limiting function y^N in the denominator unchanged, while separating out the y^N in the numerator into N distinct roots.

There is another notion of stability of an integral that should be mentioned in this context, though it is not directly related to the A_p problem. Suppose δ is a positive number such that

$$\int_{I(r)\subset\mathbb{R}^n} |f|^{-\delta} < \infty, \tag{3.9}$$

where $I(r)$ (respectively $U(r)$) denotes the closed polydisk of radius r centered at the origin in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). According to the definition of Phong, Stein and Sturm [PSS99], the integral in (3.9) is called stable for the above value of δ if there exists $0 < s < r$ and $\epsilon_0 > 0$ such that

$$\sup_g \int_{I(s)\subset\mathbb{R}^n} |g|^{-\delta} < \infty,$$

where the supremum is taken over all g holomorphic on $U(r)$ and satisfying

$$|f - g|_{U(r)} < \epsilon_0.$$

Here we have denoted, by a slight abuse of notation, the unique holomorphic extension of f to $U(r)$ by f as well. It has been shown in [PSS99] that an unweighted integral of the form (3.9) is stable in $\mathbb{R}^2$, according to the above definition, for all values of δ that ensure convergence of the integral in (3.9). In dimension 3, the integral in (3.9) has been shown to be stable provided $\delta < \frac{2}{N}$, where N is the multiplicity of f at the origin. Varchenko's counterexample (section 5 in [Var76]) shows that this result is, in general, sharp.

There exists a suitable variant of the stability problem mentioned above for the weighted integral. This is especially interesting if the weighted integral is derived from a higher dimensional unweighted integral, as in the case of Varchenko's example. The appropriate analogue of Phong and Stein's definition of stability in this situation is the following :
Let ϵ and δ be two positive numbers such that the weighted integral in $\mathbb{R}^2$ of the form

$$\int_{I(r)} \frac{|g|^{\epsilon}}{|f|^{\delta}}$$

is finite. The above integral will be called stable for the given values of ϵ and δ if there exists $0 < s < r$ and $\epsilon_0 > 0$ such that

$$\sup_h \int_{I(s)} \frac{|g|^\epsilon}{|h|^\delta} < \infty,$$

where the supremum is taken over all functions h holomorphic on $U(r)$ and satisfying $|f - h|_{U(r)} < \epsilon_0$.

Varchenko's example, coupled with Phong and Stein's results in dimensions 2 and 3, suggests that the two-dimensional weighted integral is, in general, not stable in the above sense with respect to the class of all holomorphic perturbations of f even though its unweighted analogue is. This however opens up new problems of its own, and it is natural to ask if there is a different class of perturbations for which stability does hold in the weighted situation. We shall return to these issues in a subsequent paper.

Acknowledgement : I would like to thank Prof. Michael Christ for numerous valuable discussions during the preparation of this paper.

References

[Kar86a] V.N. Karpushkin, *A theorem concerning uniform estimates of oscillatory integrals when the phase is a function of two variables*, J. Soviet Math. **35** (1986), 2809–2826.

[Kar86b] ______, *Uniform estimates of oscillatory integrals with parabolic or hyperbolic phases*, J. Soviet Math. **33** (1986), 1159–1188.

[Pra] M. Pramanik, *Convergence of two-dimensional weighted integrals*, preprint.

[PS97] D.H. Phong and E.M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Math. **179** (1997), no. 1, 105–152.

[PSS99] D.H. Phong, E.M. Stein, and J.A. Sturm, *On the growth and stability of real-analytic functions*, Amer. J. Math. **121** (1999), no. 3, 519–554.

[Ste93] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, 1993.

[Var76] A Varchenko, *Newton polyhedron and estimation of oscillating integrals*, Funct. Anal. Appl. **18** (1976), 175–196.