L^p REGULARITY OF AVERAGES OVER CURVES AND BOUNDS FOR ASSOCIATED MAXIMAL OPERATORS

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Abstract. We prove that for a finite type curve in $\mathbb{R}^3$ the maximal operator generated by dilations is bounded on L^p for sufficiently large p. We also show the endpoint $L^p \to L^p_{1/p}$ regularity result for the averaging operators for large p. The proofs make use of a deep result of Thomas Wolff about decompositions of cone multipliers.

1. Introduction and Statement of Results

Let I be a compact interval and consider a smooth curve

$$\gamma : I \to \mathbb{R}^3.$$

We say that γ is of finite type on I if there is a natural number n, and $c > 0$ so that for all $s \in I$, and for all $|\xi| = 1$,

$$\sum_{j=1}^{n} \big|\langle \gamma^{(j)}(s), \xi \rangle\big| \geq c. \tag{1.1}$$

For fixed s the smallest n for which (1.1) holds is the *type* of γ at s. The type is an upper semicontinuous function, and we refer to the supremum of the types over $s \in I$ as the maximal type of γ on I. Let χ be a smooth function supported in the interior of I. We define a measure μ_t supported on a dilate of the curve by

$$\langle \mu_t, f \rangle := \int f\,(t\gamma(s))\,\chi(s)\,ds, \tag{1.2}$$

and set

$$\mathcal{A}_t f(x) := f * \mu_t(x). \tag{1.3}$$

We are aiming to prove sharp L^p regularity properties of these integral operators and also L^p boundedness of the maximal operator given by

$$\mathcal{M}f(x) := \sup_{t>0} |\mathcal{A}_t f(x)|. \tag{1.4}$$

To the best of our knowledge, L^p boundedness of $\mathcal{M}$ had not been previously established for any $p < \infty$. Here we prove some positive results for large p and in particular answer affirmatively a question on maximal functions associated to helices which has been around for a while (for example it was explicitly formulated in a circulated but unpublished survey by Christ [4] from the late 1980's).

Our results rely on a deep inequality of Thomas Wolff for decompositions of the cone multiplier in $\mathbb{R}^3$. To describe it consider a distribution $f \in \mathcal{S}'\left(\mathbb{R}^3\right)$ whose

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Fourier transform is supported in a neighborhood of the light cone $\xi_3^2 = \xi_1^2 + \xi_2^2$ at level $\xi_3 \approx 1$, of width $\delta \ll 1$. Let $\{\Psi_\nu\}$ be a collection of smooth functions which are supported in $1 \times \delta^{1/2} \times \delta$-plates that "fit" the light cone and satisfy the natural size estimates and differentiability properties; for a more precise description see §2. Wolff [22] proved that for all sufficiently large p, say $p > p_W$, and all $\epsilon > 0$, there exists $C_{\epsilon,p} > 0$ such that

$$\Big\| \sum_\nu \widehat{\Psi}_\nu * f \Big\|_p \le C_{\epsilon,p} \delta^{-\frac{1}{2}+\frac{2}{p}-\epsilon} \Big(\sum_\nu \big\| \widehat{\Psi}_\nu * f \big\|_p^p \Big)^{\frac{1}{p}}. \tag{1.5}$$

A counterexample in [22] shows that this inequality cannot hold for all $\varepsilon > 0$ if $p < 6$ and Wolff obtained (1.5) for $p \geq 74$ (a slightly better range can be obtained as was observed by Garrigós and one of the authors [8]).

We note that connections between cone multipliers and the regularity properties of curves with nonvanishing curvature and torsion have been used in various previous papers, first implicitly in the paper by Oberlin [14] who proved sharp $L^p \to L^2$ estimates for, say, convolutions with measures on the helix $(\cos s, \sin s, s)$; these were extended in [9] to more general classes of Fourier integral operators. Concerning L^p Sobolev estimates the $L^p \to L^p_{2/3p}$ boundedness follows by an easy interpolation argument, but improvements of this estimate are highly nontrivial. Oberlin, Smith and Sogge [16] used results by Bourgain [3] and Tao and Vargas [21] on square-functions associated to cone multipliers to show that if $2 < p < \infty$ then the averages for the helix map L^p to the Sobolev space L^p_α, for some $\alpha > 2/(3p)$.

We emphasize that sharp regularity results for *hypersurfaces* have been obtained from interpolation arguments, using results on damped oscillatory integrals and an improved L^∞ bound near "flat parts" of the surface, see *e.g.* [6], [19], [10] and elsewhere. However this interpolation technique does not apply to averages over manifolds with very high codimension, in particular not to curves in $\mathbb{R}^d$, $d \geq 3$.

Our first result on finite type curves in $\mathbb{R}^3$ concerns the averaging operator $\mathcal{A} \equiv \mathcal{A}_1$ in (1.3); it depends on the optimal exponent p_W in Wolff's inequality (1.5).

Theorem 1.1. *Suppose that $\gamma \in C^{n+5}(I)$ is of maximal type n, and suppose that*

$$\max\{n, \frac{p_W + 2}{2}\} < p < \infty.$$

Then $\mathcal{A}$ maps L^p boundedly to the Sobolev space $L^p_{1/p}$.

Thus the sharp L^p-Sobolev regularity properties for the helix hold for $p > 38$, according to Wolff's result. It is known by an example due to Oberlin and Smith [15] that the $L^p \to L^p_{1/p}$ regularity result fails if $p < 4$. Recall that Wolff's inequality (1.5) is conjectured for $p \in (6, \infty)$, and thus establishing this conjecture would by Theorem 1.1 imply the $L^p \to L^p_{1/p}$ bound for $p > 4$. If the type n is sufficiently large then our result is sharp; it can be shown by a modification of an example by Christ [5] for plane curves that the endpoint $L^n \to L^n_{1/n}$ bound fails. Finally we note that by a duality argument and standard facts on Sobolev spaces one can also deduce sharp bounds near $p = 1$, namely if $1 < p < \min\{n/(n-1), (p_W + 2)/p_W\}$ then $\mathcal{A}$ maps L^p boundedly to $L^p_{1/p'}$.

Remark. There are generalizations of Theorem 1.1 which apply to variable curves; one assumes that the associated canonical relation in $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$ projects to each

$T^*\mathbb{R}^3$ only with fold singularities and that a curvature condition in [9] on the fibers of the singular set is satisfied. We intend to take up these matters in a forthcoming paper [17].

Our main result on the maximal operator $\mathcal{M}$ is

Theorem 1.2. *Suppose that $\gamma \in C^{n+5}$ is of maximal type n, then $\mathcal{M}$ defines a bounded operator on L^p for $p > \max(n, (p_W + 2)/2)$.*

Again the range of p's is only optimal if the maximal type is sufficiently large (*i.e.* $n \geq (p_W + 2)/2$). The following measure-theoretic consequence (which only uses L^p boundedness for some $p < \infty$) appears to be new; it follows from Theorem 1.2 by arguments in [2].

Corollary 1.3. *Let $\gamma : I \to \mathbb{R}^3$ be smooth and of finite type and let $A \subset \mathbb{R}^3$ be a set of positive measure. Let E be a subset of $\mathbb{R}^3$ with the property that for every $x \in A$ there is a $t(x) > 0$ such that $x + t(x)\gamma(I)$ is contained in E. Then E has positive outer measure.*

In itself the regularity result of Theorem 1.1 does not imply boundedness of the maximal operator, but a local smoothing estimate can be used. This we only formulate for the nonvanishing curvature and torsion case.

Theorem 1.4. *Suppose that $\gamma \in C^5(I)$ has nonvanishing curvature and torsion. Let $p_W < p < \infty$ and $\chi \in C_0^\infty((1,2))$. Suppose that $\alpha < 4/(3p)$. Then the operator $\mathfrak{A}$ defined by $\mathfrak{A}f(x,t) = \chi(t)\mathcal{A}_t f(x)$ maps $L^p(\mathbb{R}^3)$ boundedly into $L^p_\alpha(\mathbb{R}^4)$.*

By interpolation with the standard $L^2 \to L^2_{1/3}$ estimate (obtained from van der Corput's lemma) one sees that $\mathfrak{A}$ maps $L^p(\mathbb{R}^3)$ to $L^p_{\beta(p)}(\mathbb{R}^4)$, for some $\beta(p) > 1/p$, if $(p_W + 2)/2 < p < \infty$. By standard arguments the L^p boundedness of the maximal operator $\mathcal{M}$ follows in this range (provided that the curve has nonvanishing curvature and torsion).

Structure of the paper. In §2 we prove an extension of Wolff's estimate to general cones which will be crucial for the arguments that follow. In §3 we prove the sharp L^p Sobolev estimates for large p (Theorem 1.1). In §4 we prove a version of the local smoothing estimate for averaging operators associated to curves in $\mathbb{R}^d$ microlocalized to the nondegenerate region where $\langle \gamma''(s), \xi \rangle \neq 0$. In §5 we use the previous estimates and rescaling arguments to prove Theorem 1.4 and in §6 we deduce our results for maximal operators, including an estimate for a two parameter family of helices.

2. Variations of Wolff's inequality

The goal of this section is to prove a variant of Wolff's estimate (1.5) where the standard light-cone is replaced by a general cone with one nonvanishing principal curvature. Rather than redoing the very complicated proof of Wolff's inequality we shall use rescaling and induction on scales arguments to deduce the general result from the special result, assuming the validity of (1.5) for the light cone, in the range $p \geq p_W$.

We need to first set up appropriate notation. Let I be a closed subinterval of $[-1, 1]$ and let

$$\alpha \mapsto g(\alpha) = (g_1(\alpha), g_2(\alpha)) \in \mathbb{R}^2, \quad \alpha \in I,$$

define a C^3 curve in the plane and we assume that for positive b_0, b_1 and b_2

$$\begin{aligned} &\|g\|_{C^3(I)} \le b_0, \\ &|g'(\alpha)| \ge b_1, \\ &|g_1'(\alpha)g_2''(\alpha) - g_2'(\alpha)g_1''(\alpha)| \ge b_2. \end{aligned} \tag{2.1}$$

We consider multipliers supported near the cone

$$\mathcal{C}_g = \{\xi \in \mathbb{R}^3 : \xi = \lambda(g_1(\alpha), g_2(\alpha), 1), \quad \alpha \in I, \lambda > 0\}.$$

For each α we set

$$u_1(\alpha) = (g(\alpha), 1), \quad u_2(\alpha) = (g'(\alpha), 0), \quad u_3(\alpha) = u_1(\alpha) \times u_2(\alpha) \tag{2.2}$$

where $\times$ refers to the usual cross product so that a basis of the tangent space of $\mathcal{C}_g$ at $(g(\alpha), 1)$ is given by $\{u_1(\alpha), u_2(\alpha)\}$. Let $\lambda \ge 0$, $\delta > 0$ and define the (δ, λ)-plate at α, $R^\alpha_{\delta,\lambda}$, to be the parallelepiped in $\mathbb{R}^3$ given by the inequalities

$$\begin{aligned} &\lambda/2 \le |\langle u_1(\alpha), \xi\rangle| \le 2\lambda, \\ &|\langle u_2(\alpha), \xi - \xi_3 u_1(\alpha)\rangle| \le \lambda\delta^{1/2}, \\ &|\langle u_3(\alpha), \xi\rangle| \le \lambda\delta. \end{aligned} \tag{2.3}$$

For a constant $A \ge 1$ we define the *A-extension* of the plate $R^\alpha_{\delta,\lambda}$ to be the parallelepiped given by the inequalities

$$\begin{aligned} &\lambda/(2A) \le |\langle u_1(\alpha), \xi\rangle| \le 2A\lambda; \\ &|\langle u_2(\alpha), \xi - \xi_3 u_1(\alpha)\rangle| \le A\lambda\delta^{1/2}; \\ &|\langle u_3(\alpha), \xi\rangle| \le A\lambda\delta. \end{aligned}$$

Note that the A-extension of a (δ, λ)-plate has width $\approx A\lambda$ in the radial direction tangent to the cone, width $\approx A\lambda\delta^{1/2}$ in the tangential direction which is perpendicular to the radial direction and is supported in a neighborhood of width $\approx A\lambda\delta$ of the cone.

A C^∞ function ϕ is called an *admissible bump function associated to $R^\alpha_{\delta,\lambda}$* if ϕ is supported in $R^\alpha_{\delta,\lambda}$ and if

$$\big|\langle u_1(\alpha), \nabla\rangle^{n_1} \langle u_2(\alpha), \nabla\rangle^{n_2} \langle u_3(\alpha), \nabla\rangle^{n_3} \phi(\xi)\big| \le \lambda^{-n_1-n_2-n_3}\delta^{-n_2/2}\delta^{-n_3}, \quad 0 \le n_1 + n_2 + n_3 \le 4. \tag{2.4}$$

A C^∞ function ϕ is called an *admissible bump function associated to the A-extension of $R^\alpha_{\delta,\lambda}$* if ϕ is supported in the A-extension but still satisfies the estimates (2.4).

Let $\lambda \ge 1$, $\delta > 0$, $\delta^{1/2} \le \theta$, moreover $\sigma \le \delta^{1/2}$. A finite collection $\mathcal{R} = \{R_\nu\}_{\nu=1}^N$ is called a *$(\delta, \lambda, \theta)$-plate family with separation σ associated to g* if (i) each R_ν is of the form $R^{\alpha_\nu}_{\delta,\lambda}$ for some $\alpha_\nu \in I$, (ii) $\nu \ne \nu'$ implies that $|\alpha_\nu - \alpha_{\nu'}| \ge \sigma$ and, (iii) $\max_{R_\nu \in \mathcal{R}}\{\alpha_\nu\} - \min_{R_\nu \in \mathcal{R}}\{\alpha_\nu\} \le \theta$. Given $A \ge 1$ we let the *A-extension of the plate family $\mathcal{R}$* consist of the A-extensions of the plates R_ν.

The main result in Wolff's paper is proved for the cone generated by $g(\alpha) = (\cos 2\pi\alpha, \sin 2\pi\alpha)$, $-1/2 \le \alpha \le 1/2$. Namely if $\mathcal{R}$ is a $(\delta, \lambda, 1)$-plate family with separation $\sqrt{\delta}$ and for $R \in \mathcal{R}$, ϕ_R is an admissible bump function associated to R

then for all $\varepsilon > 0$ there is the inequality

$$\Big\| \sum_{R\in\mathcal{R}} \mathcal{F}^{-1}[\phi_R \widehat{f}_R] \Big\|_p \le A(\varepsilon)\delta^{\frac{2}{p}-\frac{1}{2}-\varepsilon}\Big(\sum_{R\in\mathcal{R}} \|f_R\|_p^p\Big)^{1/p} \tag{2.5}$$

if $p > 74$. This is equivalent with the statement (1.5) in the introduction. Our next proposition says that this inequality for the light cone implies an analogous inequality for a general curved cone.

Proposition 2.1. *Suppose that $p > 2$ and that (2.5) holds for all $(\delta, \lambda, 1)$-plate families associated to the circle $\{(\cos(2\pi\alpha), \sin(2\pi\alpha))\}$. Then for any $\delta \le 1$, $\lambda \ge 1$, $\sigma \le \sqrt{\delta}$ the following holds true.*

Let $\alpha \mapsto g(\alpha)$ satisfy (2.1) and let $\mathcal{R}$ be a $(\delta, \lambda, \theta)$-plate family with separation σ, associated to g. For each R let ϕ_R be an admissible bump function associated to R. Then for $\varepsilon > 0$ there is a constant $C(\varepsilon)$ depending only on ε and the constants b_0, b_1, b_2 in (2.1) so that for $f_R \in L^p(\mathbb{R}^3)$,

$$\Big\| \sum_{R\in\mathcal{R}} \mathcal{F}^{-1}[\phi_R \widehat{f}_R] \Big\|_p \le C(\varepsilon)\delta^{1/2}\sigma^{-1}(\delta\theta^{-2})^{\frac{2}{p}-\frac{1}{2}-\varepsilon}\Big(\sum_{R\in\mathcal{R}} \|f_R\|_p^p\Big)^{1/p}. \tag{2.6}$$

Proof. We first remark that we can immediately reduce to the case $\sigma = \sqrt{\delta}$, by a pidgeonhole argument and an application of the triangle inequality.

Secondly, if Ψ_R are bump-functions contained in the A-extensions of the rectangles R, but satisfying the same estimates (2.4) relative to the rectangles R, then an estimate such as (2.6) implies a similar estimate for the collection of bump functions $\{\Psi_R\}$ where the constant $C(\varepsilon)$ is replaced with $C_A C(\varepsilon)$. This observation will be used extensively; it is proved by a pidgeonhole and partition of unity argument.

We now use various scaling arguments based on the formula

$$\mathcal{F}^{-1}[m(L\cdot)\widehat{f}](x) = \mathcal{F}^{-1}[m\widehat{f(L^*\cdot)}](L^{*-1}x) \tag{2.7}$$

for any real invertible linear transformation L (with transpose L^*).

Step 1. Here we still assume that $g(\alpha) = (\cos 2\pi\alpha, \sin 2\pi\alpha)$, but wish to show for $\delta^{1/2} \le \theta \le 1$ the improved estimate for a $(\delta, \lambda, \theta)$-plate family with separation $\delta^{1/2}$. The plates in $\mathcal{R} \equiv \{R_\nu\}_{\nu=1}^N$ are of the form $R_\nu = R_{\delta,\lambda}^{\alpha_\nu}$ where $|\alpha_\nu - \alpha^0| \le \theta$ for some $\alpha^0 \in [0, 1]$. Let L_1 be the rotation by the angle α in the (ξ_1, ξ_2) plane which leaves the ξ_3 axis fixed. Then the family the plate family $L_1(R_\nu)$ is still a $(\delta, \lambda, \theta)$ plate family with associated bump functions $\phi_{\nu,1} = \phi_{R_\nu} \circ L_1^{-1}$. We note that all rotated plates are contained in a larger $(C_1\lambda, C_1\lambda\theta, C_1\lambda\theta^2)$ rectangle with axes in the direction of $(1, 0, 1)$, $(0, 1, 0)$, $(1, 0, -1)$.

We now use a rescaling argument from [21] and [22]. Let L_2 be the linear transformation that maps $(1, 0, 1)$ to $(1, 0, 1)$, $(0, 1, 0)$ to $\theta^{-1}(0, 1, 0)$ and $(1, 0, -1)$ to $\theta^{-2}(1, 0, -1)$; it leaves the light cone invariant. One checks that each parallelepiped $L_2 \circ L_1 R_\nu$ is contained in a $(C_2\lambda, C_2\lambda\delta^{1/2}\theta^{-1}, C_2\lambda\delta\theta^{-2})$ plate $\widetilde{R}_\nu$ and the sets $\widetilde{R}_\nu$ form a C_2 extension of a $(\delta\theta^{-2}, \lambda, 1)$ family with separation $\sigma = C_3^{-1}\delta^{1/2}\theta^{-1}$. Thus using (2.7) we may apply the assumed result for $\theta = 1$, and obtain the claimed result for $\theta < 1$ (yet for the light cone).

Step 2. We shall now consider tilted cones where g is given by

$$g(\alpha) = (a + \rho\cos\alpha, b + \rho\sin\alpha), \qquad |a| + |b| + \rho + 1/\rho \le K \tag{2.8}$$

Suppose that we are given a (δ,λ,θ)-plate family $\mathcal{R}=\{R_\nu\}$ associated to g, with separation $\sqrt{\delta}$; moreover we are given a family of admissible bump-functions ϕ_ν associated with the plates R_ν. Consider the linear transformation L given by

$$L(\xi)=\Xi, \quad \text{with } \Xi_1=\frac{\xi_1-a\xi_3}{\rho}, \quad \Xi_2=\frac{\xi_2-b\xi_3}{\rho}, \quad \Xi_3=\xi_3.$$

Then the parallelepipeds $L(R_\nu)$ are contained in parallelepipeds $\widetilde{R}_\nu$ which for a suitable constant C_4 form a C_4-extension of a (δ,λ,θ)-plate family associated to the unit circle; moreover, for suitable C_5 the functions $C_5^{-1}\phi_\nu\circ L^{-1}$ form an admissible collection of bump functions associated to this extension. Here C_4, C_5 depend only on the constant K in (2.8). By scaling we obtain then estimate (2.6) for g as in (2.8), with $C(\varepsilon)$ equal to $C(K)A(\varepsilon)$.

Step 3. We now use an induction on scales argument. Let $\beta>(1/2-2/p)$ and let $W(\beta)$ denote the statement that the inequality

$$\Big\|\sum_{R\in\mathcal{R}}\mathcal{F}^{-1}[\phi_R\widehat{f}_R]\Big\|_p\le B(\beta)(\delta\theta^{-2})^{-\beta}\Big(\sum_{R\in\mathcal{R}}\|f_R\|_p^p\Big)^{1/p}. \tag{2.9}$$

holds for all g satisfying (2.1), all $\lambda>0$, $\delta\le 1$, $\sqrt{\delta}\le\theta\le 1$, all (δ,λ,θ)-plate-families associated to such g.

We remark that clearly $W(\beta)$ holds with $\beta=1$, with $B(1)$ depending only on the constants in (2.1). We shall now show that for $\beta>1/2-2/p$

$$W(\beta)\implies W(\beta') \quad \text{with} \quad \beta'=\tfrac{2}{3}\beta+\tfrac{1}{3}(\tfrac12-\tfrac{2}{p}+\varepsilon), \quad B(\beta')=C_6B(\beta)A(\varepsilon), \tag{2.10}$$

where C_6 depends only on (2.1).

In order to show (2.10) we let $\mathcal{R}=\{R^{\alpha_\nu}_{\lambda,\delta}\}$ be a (δ,λ,θ)-plate family, with associated family of bump functions $\{\phi_\nu\}$. We regroup the indices ν into families J_μ, so that for $\nu,\nu'\in J_\mu$ we have $|\alpha_\nu-\alpha_{\nu'}|\le\delta^{1/3}$ and for $\nu\in J_\mu$, $\nu'\in J_{\mu+2}$ we have $\alpha_{\nu'}-\alpha_\nu\ge\delta^{1/3}/2$. For each μ we pick one $\nu(\mu)\in J_\mu$. Then for all $\nu\in J_\mu$ the $R^{\alpha_\nu}_{\lambda,\delta}$ are contained in the C_7-extension R'_μ of a $(\delta^{2/3},\lambda)$-plate at $\alpha_\nu(\mu)$, as can be verified by a Taylor expansion. Let R''_μ be the $2C_7$ extension of that plate. We may pick a C^∞-function Ψ_μ supported in R''_μ which equals 1 on R'_μ, so that for suitable C_8 depending only on the constants in (2.1), the functions $C_8^{-1}\Psi_\mu$ are admissible bump functions associated to the R''_μ. We then use assumption $W(\beta)$ to conclude that

$$\begin{aligned}\Big\|\sum_\nu\mathcal{F}^{-1}[\phi_\nu\widehat{f}_\nu]\Big\|_p&=\Big\|\sum_\mu\sum_{\nu\in J_\mu}\mathcal{F}^{-1}[\Psi_\mu\phi_\nu\widehat{f}_\nu]\Big\|_p\\ &\le C_8B(\beta)(\delta^{2/3}\theta^{-2})^{-\beta}\Big(\sum_\mu\Big\|\sum_{\nu\in J_\mu}\mathcal{F}^{-1}[\phi_\nu\widehat{f}_\nu]\Big\|_p^p\Big)^{1/p}.\end{aligned} \tag{2.11}$$

We claim that for each μ,

$$\Big\|\sum_{\nu\in J_\mu}\mathcal{F}^{-1}[\phi_\nu\widehat{f}_\nu]\Big\|_p\le C_9A(\varepsilon)(\delta^{1/3})^{2/p-1/2-\varepsilon}\Big(\sum_{\nu\in J_\mu}\|f_\nu\|_p^p\Big)^{1/p}. \tag{2.12}$$

Clearly a combination of (2.11) and (2.12) yields (2.10) with $C_6=C_8C_9$.

We fix $\alpha'_\mu := \alpha_{\nu(\mu)}$ and observe that on the interval $[\alpha'_\mu - \delta^{1/3}, \alpha'_\mu + \delta^{1/3}]$ we may approximate the curve $\alpha \to g(\alpha)$ by its osculating circle with accuracy $\leq C_{10}\delta$. The circle is given by

$$g_\mu(\alpha) = g(\alpha'_\mu) + \rho n(\alpha'_\mu) + \rho\big(\cos(\tfrac{\alpha-\alpha'_\mu-\varphi_\mu}{\rho}), \sin(\tfrac{\alpha-\alpha'_\mu-\varphi_\mu}{\rho})\big)$$

where $n(\alpha)$ is the unit normal vector $(-g'_2(\alpha), g'_1(\alpha))/|g'(\alpha)|$, ρ is the reciprocal of the curvature of g at α'_μ and φ_μ is the unique value between 0 and 2π for which $|g'(\alpha'_\mu)|\sin(\varphi_\mu/\rho) = g'_1(\alpha'_\mu)$ and $|g'(\alpha'_\mu)|\cos(\varphi_\mu/\rho) = g'_2(\alpha_\mu)$.

In view of the good approximation property we see that for each $\nu \in J_\mu$ the plate $R^{\alpha_\nu}_{\delta,\lambda}$ associated to g is contained in the C_{11}-extension $\widetilde{R}^\nu$ of a plate $\widetilde{R}^{\alpha_\nu}_{\delta,\lambda}$ associated to g_μ. Moreover the family J_μ can be split into no more than C_{12} subfamilies J^i_μ where the α_ν in each subfamily are $\sqrt{\delta}$-separated. Finally there is C_{13} so that each bump function ϕ_ν is the C_{13}-multiple of an admissible bump function associated to $\widetilde{R}^\nu$. Here C_{11}, C_{12}, C_{13} depend only on the constants in (2.1). This puts us in the position to apply the result from step 2, with $\theta = C_{14}\delta^{1/3}$; we observe that the constant K in step 2 controlling in particular the radius of curvature depends again only on the constants in (2.1). Thus we can deduce (2.12) and the proof of (2.10) is complete.

Step 4. We now iterate (2.10) and replace ε by $\varepsilon/2$ to obtain (2.9) with

$$\beta \equiv \beta_n = (\tfrac{2}{3})^n + (1-(\tfrac{2}{3})^n)(\tfrac{1}{2} - \tfrac{2}{p} + \tfrac{\varepsilon}{2})$$
$$B(\beta_n) = \big(C_{15}A(\tfrac{\varepsilon}{2})\big)^n, \quad n = 1, 2, \dots$$

The conclusion of the proposition follows if we choose $n > \log(2/\varepsilon)/\log(3/2)$. $\square$

One can use Proposition 2.1 and standard arguments to see that results on the circular cone multiplier in [22] carry over to more general cones. To formulate such a result let $\rho \in C^4(\mathbb{R}^2 \setminus \{0\})$ be positive away from the origin, and homogeneous of degree 1. Consider the Fourier multiplier in $\mathbb{R}^3$, given by

$$m_\lambda(\xi', \xi_3) = (1 - \rho(\xi'/\xi_3))^\lambda_+.$$

As in [22] we obtain

Corollary 2.2. *Assume that the unit sphere $\Sigma_\rho = \{\xi' \in \mathbb{R}^2 : \rho(\xi') = 1\}$ has nonvanishing curvature everywhere. Then m_λ is a Fourier multiplier of $L^p(\mathbb{R}^3)$ if $\lambda > (1/2 - 2/p)$, $p_W \leq p < \infty$.*

Remarks.

(i) The curvature condition on Σ_ρ in the corollary can be relaxed by scaling arguments.

(ii) The methods of Proposition 2.1 apply in higher dimensions as well. In particular they generalize Wolff's inequality for decompositions of light cones in higher dimensions ([12]) to more general elliptical cones generated by convex hypersurfaces with nonvanishing curvature. In particular, if ρ is a sufficiently smooth distance function in $\mathbb{R}^d$, $m_\lambda(\xi', \xi_{d+1}) = (1 - \rho(\xi'/|\xi_{d+1}|))^\lambda_+$ and if the unit sphere associated with ρ is a convex hypersurface of $\mathbb{R}^d$ with nonvanishing Gaussian curvature then m_λ is a Fourier multiplier of $L^p(\mathbb{R}^{d+1})$, for $\lambda > d|1/2 - 1/p| - 1/2$, for the range of p's given in [12] for the multipliers associated with the spherical cone. After

Proposition 2.1 had been obtained Łaba and Pramanik [11] worked out an alternative proof of the higher dimensional variant directly based on the methods in [22], [12]. Their approach also applies to the case of nonelliptical cones which cannot be obtained by scaling and approximation from existing results.

3. L^p regularity

We shall first consider a "nondegenerate case", namely we assume that $s \mapsto \gamma(s) \in \mathbb{R}^3$, $s \in I \subset [-1,1]$ is of class C^5 and has *nonvanishing curvature and torsion.* We assume that

$$\sum_{i=1}^{5} |\gamma^{(i)}(s)| \le C_0, \qquad s \in I \tag{3.1}$$

and

$$\Big|\det\big(\gamma'(s) \quad \gamma''(s) \quad \gamma'''(s)\big)\Big| \ge c_0, \qquad s \in I \tag{3.2}$$

In this case we show Theorem 1.1 under the assumption that the cutoff function χ in (1.2) is of class C^4; then we may without loss of generalization assume that γ is parametrized by arclength (since reparametrization introduces just a different C^4 cutoff). In the end of this section we shall describe how to extend the result to the finite type case.

In what follows we shall write $\mathcal{E}_1 \lesssim \mathcal{E}_2$ for two quantities $\mathcal{E}_1$, $\mathcal{E}_2$ if $\mathcal{E}_1 \le C\mathcal{E}_2$ with a constant C only depending on the constants in (3.1), (3.2). We denote by $T(s), N(s), B(s)$ the Frenet frame of unit tangent, unit normal and unit binormal vector. We recall the Frenet equations $T' = \kappa N$, $N' = -\kappa T + \tau B$, $B' = -\tau N$, with curvature κ and τ . The assumption of nonvanishing curvature and torsion implies that the cone generated by the binormals, $\mathfrak{B} = \{rB(s) : r > 0, s \in I\}$, has one nonvanishing principal curvature which is equal to $r\kappa(s)\tau(s)$ at $\xi = rB(s)$.

By localization in s and possible rotation we may assume for the third component of $B(s)$ that $B_3(s) > 1/2$ for all $s \in I$. If

$$g(s) = \big(\tfrac{B_1(s)}{B_3(s)}, \tfrac{B_2(s)}{B_3(s)}\big) \tag{3.3}$$

parametrizes the level curve at height $\xi_3 = 1$ then the curvature property of the cone can be expressed in terms of the curvature of this level curve and a computation gives

$$\det\begin{pmatrix} g_1'(s) & g_2'(s) \\ g_1''(s) & g_2''(s) \end{pmatrix} = \frac{1}{(B_3(s))^3} \det\begin{pmatrix} B_1'(s) & B_2'(s) & B_3'(s) \\ B_1''(s) & B_2''(s) & B_3''(s) \\ B_1(s) & B_2(s) & B_3(s) \end{pmatrix} = \frac{\kappa(s)\tau(s)}{(B_3(s))^3}.$$

Thus the hypotheses on g in (2.1) are satisfied with constants depending only on the constants in (3.1), (3.2).

We shall work with standard Littlewood-Paley cutoffs, and make decompositions of the Fourier multiplier associated to the averages. Observe first that the contribution of the multiplier near the origin is irrelevant in view of the compact support of the kernel. Thus consider for $k > 0$ the Fourier multipliers

$$m_k(\xi) = \int e^{-it\langle\gamma(s),\xi\rangle} a_k(s, 2^{-k}\xi)ds \tag{3.4}$$

where we assume that a_k vanishes outside the annulus $\{\xi : 1/2 < |\xi| < 2\}$ and satisfies the estimates

$$|\partial_s^j \partial_\xi^\alpha a(s,\xi)| \le C_2, \quad |\alpha| \le 2, 0 \le j \le 3; \tag{3.5}$$

here of course $|\alpha| = |\alpha_1|+|\alpha_2|+|\alpha_3|$. Thus the multiplier m_k is a symbol of order 0, with perhaps limited order of differentiability, localized to the annulus $\{|\xi| \approx 2^k\})$. We note that by the standard Bernstein theorem (which says that $L^2_\alpha \subset \mathcal{F}[L^1]$ for $\alpha > 3/2$) the multipliers $a_k(s,\cdot)$ and their s-derivatives up to order three are Fourier multipliers of $L^p(\mathbb{R}^3)$, uniformly in s, k.

We have to establish that for the desired range of p's the sum $\sum_{k>0} 2^{k/p} m_k$ is a Fourier multiplier of L^p. We may assume that the symbols a_k are supported near from the cone generated by the binormal vectors $B(s)$. More precisely if $\theta(\xi)$ is smooth away from the origin and homogeneous of degree 0 and if θ has the property that

$$|\langle \gamma''(s), \tfrac{\xi}{|\xi|}\rangle| \ge c > 0, \qquad \xi \in \operatorname{supp}(\theta) \cap \operatorname{supp}(a_k),$$

then $\|\theta m_k\|_\infty = O(2^{-k/2})$ by van der Corput's Lemma, and by the almost disjointness of the supports we also have that $\|\theta \sum_{k>0} 2^{k/2} m_k\|_\infty = O(1)$ by van der Corput's Lemma. Moreover by standard singular integral theory the operator with Fourier multiplier $\theta \sum_{k>0} m_k$ maps L^∞ to BMO and consequently, by analytic interpolation $\theta \sum_{k>0} 2^{k/p} m_k$ is a Fourier multiplier of L^p provided $2 \le p < \infty$.

Thus by a partition of unity it suffices to understand the localization of the multiplier $\sum_{k>0} 2^{k/p} m_k$ to a narrow (tubular) neighborhood of the binormal cone $\mathfrak{B} = \{rB(s) : r > 0, s \in I\}$, and therefore in what follows we may and shall assume that ξ in the support of $a_k(s,\cdot)$ can be expressed as

$$\xi = rB(\sigma) + uT(\sigma) =: \Xi(r,u,\sigma),$$

with inverse function $\xi \mapsto (r(\xi), u(\xi), \sigma(\xi))$.

Decomposition of the dyadic multipliers. We shall now concentrate on the multipliers m_k in (3.4), and prove the bound $\|m_k\|_{M_p} \lesssim 2^{-k/p}$, for $p > (p_W+2)/2$. Here M_p is the usual Fourier multiplier space.

We first decompose further our symbols a_k. Let $\eta_0 \in C_0^\infty(\mathbb{R})$ be an even function supported in $[-1,1]$ and be equal to 1 on $[-1/2,1/2]$. Let $\eta_1 = \eta_0(4^{-1}\cdot) - \eta_0$. Let $A_0 \gg 2\max\{1, 1/\tau(s) : s \in I\}$ and set

$$\widetilde{a}_k(s,\xi) = a_k(s,\xi)\eta_0(2^{2[k/3]}(|u(\xi)| + (s-\sigma(\xi))^2)), \tag{3.6}$$

and for integers $l < k/3$

$$\begin{aligned} a_{k,l}(s,\xi) &= a_k(s,\xi)\, \eta_1(2^{2l}(|u(\xi)| + (s-\sigma(\xi))^2))\, \eta_0(\tfrac{(s-\sigma(\xi))^2}{A_0 u(\xi)}) \\ b_{k,l}(s,\xi) &= a_k(s,\xi)\, \eta_1(2^{2l}(|u(\xi)| + (s-\sigma(\xi))^2))\, \big(1 - \eta_0(\tfrac{(s-\sigma(\xi))^2}{A_0 u(\xi)})\big). \end{aligned} \tag{3.7}$$

Thus $a_{k,l}(s,\cdot)$ is supported where $\operatorname{dist}(\xi,\mathfrak{B}) \approx 2^{-2l}$ and $|s-\sigma(\xi)| \lesssim 2^{-l}$, and $\widetilde{a}_k(s,\cdot)$ is supported in a $C2^{-2k/3}$ neighborhood of the binormal cone with $|s-\sigma(\xi)| \lesssim 2^{-k/3}$. The symbol $b_{k,l}(s,\cdot)$ is supported in a $C2^{-2l}$ neighborhood of the binormal cone but now $|s-\sigma(\xi)| \approx 2^{-l}$.

We note that in view of the preliminary localizations the symbols $a_{k,l}$, $b_{k,l}$ vanish for $l \le C$, moreover

$$a_k(s,\xi) = \widetilde{a}_k(s,\xi) + \sum_{l \le k/3} a_{k,l}(s,\xi) + \sum_{l \le k/3} b_{k,l}(s,\xi).$$

Set

$$m_k[a](\xi) = \int a(s, 2^k\xi) e^{-i\langle \gamma(s), \xi\rangle} ds. \tag{3.8}$$

We shall show

Proposition 3.1. *For $p_W < p < \infty$,*

$$\|m_k[\widetilde{a}_k]\|_{M^p} \le C_\varepsilon 2^{-4k/3p + k\varepsilon}, \tag{3.9}$$

$$\|m_k[a_{k,l}]\|_{M^p} \le C_\varepsilon 2^{-k/p} 2^{-l/p + l\varepsilon}, \tag{3.10}$$

$$\|m_k[b_{k,l}]\|_{M^p} \le C_\varepsilon 2^{-2k/p} 2^{2l/p + l\varepsilon}, \tag{3.11}$$

The constants depend only on ε, (3.1)*,* (3.2) *and* (3.5)*.*

We shall give the proof of (3.10) and (3.11), and the proof of (3.9) is are analogous with mainly notational changes.

For the proofs of (3.10) and (3.11) we need to further split the symbols $a_{k,l}$, $b_{k,l}$ by making an equally spaced decomposition into pieces supported on 2^{-l} intervals. Let $\zeta \in C_0^\infty$ be supported in $(-1, 1)$ so that $\sum_{\nu \in \mathbb{Z}} \zeta(\cdot - \nu) \equiv 1$. We set

$$a_{k,l,\nu}(s, \xi) = \zeta(2^l s - \nu) a_{k,l}(s, \xi)$$

and similarly $b_{k,l,\nu}(s, \xi) = \zeta(2^l s - \nu) b_{k,l}(s, \xi)$; moreover define $\widetilde{a}_{k,\nu}(s, \xi) = \zeta(2^{k/3}s - \nu)\widetilde{a}_k(s, \xi)$.

In order to apply Wolff's estimate in the form of Proposition 2.1 we need

Lemma 3.2. *Let $s_\nu = 2^{-l}\nu$ and let $(T_\nu, N_\nu, B_\nu) = (T(s_\nu), N(s_\nu), B(s_\nu))$. Suppose that $|s - s_\nu| \le 2^{2-l}$. Then the following holds true:*

(i) The multipliers $a_{k,l,\nu}(s, \cdot)$, $b_{k,l,\nu}(s, \cdot)$ are supported in

$$\{\xi : |\langle \xi, T_\nu\rangle| \le C2^{-2l}, |\langle \xi, N_\nu\rangle| \le C2^{-l}, C^{-1} \le |\langle \xi, B_\nu\rangle| \le C\} \tag{3.12}$$

where C only depends on the constants in (3.1)*.*

(ii) For $j = 0, 1, 2,$ and $h_\nu = a_{k,l,\nu}(s, \cdot)$ or $b_{k,l,\nu}(s, \cdot)$

$$\left|\big(\langle T_\nu, \nabla\rangle\big)^j h_\nu\right| \le C' 2^{2lj}, \tag{3.13}$$

$$\left|\big(\langle N_\nu, \nabla\rangle\big)^j h_\nu\right| \le C' 2^{lj}, \tag{3.14}$$

$$\left|\big(\langle B_\nu, \nabla\rangle\big)^j h_\nu\right| \le C'. \tag{3.15}$$

(iii) The statements analogous to (i), (ii) hold true for $\widetilde{a}_{k,\nu}(s, \cdot)$, $\widetilde{b}_{k,\nu}(s, \cdot)$, with l replaced by $[k/3] + 1$.

(iv) If h_ν is any of the multipliers $a_{k,l,\nu}(s, \cdot)$, $b_{k,l,\nu}(s, \cdot)$, then the statements analogous to (i)-(iii) hold for the multiplier $h_\nu(\xi) 2^l \langle \gamma''(s), \xi\rangle$. Similarly, if $\widetilde{h}_\nu$ denotes any of $\widetilde{a}_{k,\nu}(s, \cdot)$ or $\widetilde{b}_{k,\nu}(s, \cdot)$ then $\widetilde{h}_\nu$ can be replaced by $\widetilde{h}_\nu 2^l \langle \gamma''(s), \xi\rangle$.

Proof. To see the containment of $\operatorname{supp} a_{k,l,\nu}(s, \cdot)$ in the set (3.12) we assume that $\xi = \Xi(r, u, \sigma)$ and expand $B(\sigma)$, $T(\sigma)$ about $\sigma = s_\nu$. Using the Frenet formulas for ξ in the support of $a_{k,l,\nu}(s, \cdot)$ we obtain

$$\langle \xi, T_\nu\rangle = \langle rB(\sigma) + uT(\sigma), T_\nu\rangle = O((\sigma - s_\nu)^2) + O(u) = O(2^{-2l})$$

and similarly $\langle \xi, N_\nu\rangle = O(2^{-l})$.

To show (3.13) we use the formulas

$$\nabla r = B, \quad \nabla u = T, \quad \nabla\sigma = \frac{1}{u\kappa - r\tau} N;$$

here of course $r = r(\xi)$, $B = B(\sigma(\xi))$, etc. Moreover

$$\begin{aligned}\langle e, \nabla\rangle^2 r &= \frac{-\tau}{-r\tau + u\kappa}\langle e, N\rangle^2, \\ \langle e, \nabla\rangle^2 u &= \frac{1}{-r\tau + u\kappa}\langle e, N\rangle\langle e, T\rangle, \\ \langle e, \nabla\rangle^2 \sigma &= \frac{1}{-r\tau + u\kappa}\langle e, N\rangle\langle e, \nabla(\tfrac{1}{-r\tau+u\kappa})\rangle.\end{aligned}$$

From these formulas and the chain rule the verification of the asserted differentiability properties is straightforward; we use also that $T_\nu - T(\sigma(\xi)) = O(2^{-l})$ and similar statements for N_ν and B_ν. $\square$

We shall need bounds for the L^1 and L^2 operator norms of the operators defined by

$$\widehat{\mathcal{A}^{k,l,\nu} f}(\xi) = m_k[a_{k,l,\nu}]\widehat{f}(\xi), \quad \widehat{\widetilde{\mathcal{A}}^{k,\nu} f}(\xi) = m_k[\widetilde{a}_{k,\nu}]\widehat{f}(\xi), \tag{3.16}$$

and

$$\widehat{\mathcal{B}^{k,l,\nu} f}(\xi) = m_k[b_{k,l,\nu}]\widehat{f}(\xi). \tag{3.17}$$

We remark that part (iii) of the following lemma (and also part (iv) of Lemma 3.2 above) is not needed in this section but will be needed in a proof of Theorem 6.1.

Lemma 3.3. *(i)*

$$\|\mathcal{A}^{k,l,\nu}\|_{L^2\to L^2} \le C2^{(l-k)/2}, \quad l \le k/3, \tag{3.18}$$

$$\|\widetilde{\mathcal{A}}^{k,\nu}\|_{L^2\to L^2} \le C2^{-k/3}, \tag{3.19}$$

$$\|\mathcal{B}^{k,l,\nu}\|_{L^2\to L^2} \le C2^{2l-k}, \quad l \le k/3. \tag{3.20}$$

(ii)

$$\|\mathcal{A}^{k,l,\nu}\|_{L^\infty\to L^\infty} + \|\mathcal{B}^{k,l,\nu}\|_{L^\infty\to L^\infty} \le C2^{-l}, \quad l \le k/3, \tag{3.21}$$

$$\|\widetilde{\mathcal{A}}^{k,\nu}\|_{L^\infty\to L^\infty} \le C2^{-k/3}. \tag{3.22}$$

(iii) Assume now that the number of sign changes of the function $s \mapsto \langle\gamma'''(s), \xi\rangle$ is bounded independent of ξ. Then the estimates in (i), (ii) continue to hold true if we replace in the above definitions any of the symbols $h_\nu = a_{k,l,\nu}(s,\cdot)$ or $b_{k,l,\nu}(s,\cdot)$ with $h_\nu 2^l\langle\gamma''(s),\xi\rangle$, or if we replace $\widetilde{h}_\nu = \widetilde{a}_{k,\nu}(s,\cdot)$ with $\widetilde{h}_\nu 2^{k/3}\langle\gamma''(s),\xi\rangle$.

Proof. The L^2 estimates (3.19) are immediate from van der Corput's lemma with third derivatives; we use that $\langle\gamma'''(s), 2^k\xi\rangle \approx 2^k$ for small $u(\xi)$. We use van der Corput's estimate for (3.18) as well and observe that for $\xi \in \operatorname{supp} a_{k,l,\nu}$ we have that $\langle\gamma''(s),\xi\rangle = (s-\sigma(\xi))\langle\gamma'''(s),\xi\rangle + O(2^{-2l})$ so that $\langle\gamma''(s), 2^k\xi\rangle \approx 2^{k-l}$ if $|s-\sigma(\xi)| \ge c_0 2^{-l}$. If c_0 is sufficiently small then we also have for $|s-\sigma(\xi)| \le c_0 2^{-l}$ that $\langle\gamma'(s),\xi\rangle = \langle\gamma'(\sigma(\xi)),\xi\rangle + O(c_0^2 2^{-2l})$ and since $|\langle\gamma'(\sigma(\xi)),\xi\rangle| = |u(\xi)|$ we get $\langle\gamma'(s), 2^k\xi\rangle \approx 2^{k-2l}$ if $|s-\sigma(\xi)| \le c_0 2^{-l}$. Thus van der Corput's lemma with one or two derivatives yields the bound

$$\big\|m_k[a_{k,l,\nu}]\big\|_\infty \le C(2^{(l-k)/2} + 2^{2l-k}) \le C'2^{(l-k)/2},$$

since $l \le k/3$.

A similar argument goes through for $m_k[b_{k,l,\nu}]$. Now $|u(\xi)| \ll |s-\sigma(\xi)| \approx 2^{-l}$ so that in the support of $b_{k,l,\nu}$ there is the lower bound $|\langle\gamma'(s), 2^k\xi\rangle| \geq c2^{k-2l}$ and van der Corput's lemma with one derivative yields

$$\|m_k[b_{k,l,\nu}]\|_\infty \lesssim 2^{2l-k}$$

and thus the asserted L^2 bound (3.20) .

We now turn to the L^∞ bounds. Consider first the multiplier $a_{k,l,\nu}$. Let L_ν be the rotation that maps the coordinate vector e_1 to T_ν, e_2 to N_ν and e_3 to B_ν. Let δ_l denote the nonisotropic dilation defined by $\delta_l(\xi) = (2^{-2l}\xi_1, 2^{-l}\xi_2, \xi_3)$. By scaling we see from (3.12) and (3.13) that $a_{k,l,\nu}(L_\nu\delta_l\cdot)$ is supported on a ball of radius C and that the directional derivatives up to order 2 in the e_1, e_2, e_3 directions are bounded, uniformly in k, l, ν, s. Thus we may apply Bernstein's theorem (alluded to above after formula (3.5)) and we see that the L^1 norms of the functions $\mathcal{F}^{-1}[a_{k,l,\nu}(s, L_\nu\delta_l\cdot)]$ are uniformly bounded. By scaling and translation we also see that the L^1 norms of the functions $\mathcal{F}^{-1}[a_{k,l,\nu}(s, 2^k\cdot)e^{i\langle\gamma(s),\cdot\rangle}]$ are uniformly bounded and thus

$$\big\|\mathcal{F}^{-1}\{m_k[a_{k,l,\nu}]\}\big\|_1 \leq \int_{|s-s_\nu|\leq 2^{2-l}} \big\|\mathcal{F}^{-1}[a_{k,l,\nu}(s, 2^k\cdot)e^{i\langle\gamma(s),\cdot\rangle}]\big\|_1 ds \leq C2^{-l}.$$

This implies the claimed L^∞ bound for $\mathcal{A}^{k,l,\nu}$. The other estimates in (ii) are obtained in the same way.

Finally we examine the statement in (iii). We note for the L^2 bounds that $\langle\gamma''(s), \xi\rangle = O(2^{-l})$ in the support of $a_{k,l,\nu}$; moreover for $|\xi| \approx 1$ the integral $\int |\langle\gamma'''(s), \xi\rangle| ds$ (over the support of the relevant cutoff function) is also $O(2^{-l})$, by an application of the fundamental theorem of calculus to a bounded number of intervals on which $\langle\gamma'''(s), \xi\rangle$ has constant sign. This estimate is needed for the application of van der Corput's lemma as before where we now gain a factor of 2^{-l}. A quick examination of the argument in Lemma 3.2 gives the claimed L^∞ bounds for this case. □

Proof of Proposition 3.1. We prove (3.10). Observe $m_k[a_{k,l}] = \sum_\nu m_k[a_{k,l,\nu}]$ where the multipliers $m_k[a_{k,l,\nu}]$ are supported in C-extensions of $(2^k, 2^{-2l})$ plates associated to the cone generated by $g(s)$ as in (3.3). This family of plates is a union of a bounded number of $c2^{-l}$ separated plate families. Consequently we can apply Wolff's estimate in the form of Proposition 2.1 and we get for $p > p_W$

$$\Big\|\sum_\nu \mathcal{A}^{k,l,\nu} f\Big\|_p \leq C_\varepsilon 2^{2l(\frac12-\frac2p+\varepsilon)}\Big(\sum_\nu \big\|\mathcal{A}^{k,l,\nu} f\big\|_p^p\Big)^{1/p} \tag{3.23}$$

Next we claim that for $2 \leq p \leq \infty$

$$\Big(\sum_\nu \big\|\mathcal{A}^{k,l,\nu} f\big\|_p^p\Big)^{1/p} \leq C2^{-l(1-3/p)}2^{-k/p}\|f\|_p \tag{3.24}$$

where for $p = \infty$ we read the left hand side as an $\ell^\infty(L^\infty)$ norm. The case for $p = \infty$ follows from (3.21) and the case $p = 2$ follows from (3.18) if we also use the finite overlap of the supports of the multipliers $m_k[a_{k,l,\nu}]$. The case for $2 < p < \infty$ follows by interpolation. Now the desired bound (3.10) follows from (3.23) and (3.24).

The estimate (3.23) holds still true if we replace $\mathcal{A}^{k,l,\nu}$ by $\mathcal{B}^{k,l,\nu}$. Moreover the argument leading to (3.24) equally applies, except that we now have a better L^2

bound $O(2^{2l-k})$ and consequently the $\ell^p(L^p)$ bound improves to

$$\Big(\sum_\nu \big\|\mathcal{B}^{k,l,\nu} f\big\|_p^p\Big)^{1/p} \le C2^{-l(1-6/p)}2^{-2k/p}\|f\|_p.$$

This yields (3.11) and the proof of the bound (3.9) is analogous. □

By a further interpolation we also obtain

Corollary 3.4. *For $p > (p_W + 2)/2$ there is $\varepsilon_0 = \varepsilon_0(p) > 0$ so that*

$$\|m_k[a_{k,l}]\|_{M^p} \le C_p 2^{-k/p}2^{-\varepsilon_0 l/p}, \tag{3.25}$$

$$\|m_k[\widetilde{a}_k]\|_{M^p} \le C_p 2^{-k(1+\varepsilon_0)/p}. \tag{3.26}$$

Moreover

$$\sum_{k\ge 3l} 2^{k/p}\|m_k[b_{k,l}]\|_{M^p} \le C_p, \tag{3.27}$$

$$\sum_k 2^{k/p}\|m_k[\widetilde{b}_k]\|_{M^p} \le C_p. \tag{3.28}$$

Proof. By the almost disjointness of our plate families and the L^2 bounds in Lemma 3.3 we see that

$$\|m_k[a_{k,l}]\|_\infty \le C2^{(l-k)/2}, \qquad \|m_k[\widetilde{a}_k]\|_\infty = O(2^{-k/3}) \tag{3.29}$$

and, similarly, $\|m_k[b_{k,l}]\|_\infty = O(2^{(l-k)/2}$ and $\|m_k[\widetilde{b}_k]\|_\infty = O(2^{-k/3})$. Interpolating the resulting L^2 estimates with the L^p bounds of Proposition 3.1 yields the assertion. □

Sobolev estimates. In order to prove Theorem 1.1 we will still have to put the estimates (3.25) for different k together. The desired estimates for the corresponding expressions involving $m[\widetilde{a}_k]$, $m[b_{k,l}]$ and $m[\widetilde{b}_k]$ follow of course from Corollary 3.4. To finish the proof of Theorem 1.1 for the case of nonvanishing curvature and torsion it suffices to show that

$$\Big\|\sum_{k\ge 3l} 2^{k/p} m[a_{k,l}]\Big\|_{M^p} \le C_p 2^{-l\varepsilon_1(p)}$$

with $\varepsilon_1(p) > 0$ if $p > (p_W + 2)/2$. In what follows we define the operator $\mathcal{A}^{k,l}$ by $\widehat{\mathcal{A}^{k,l} f} = m[a_{k,l}]\widehat{f}$.

By Littlewood-Paley theory it is sufficient to prove the vector-valued inequality

$$\Big\|\Big(\sum_{k\,:\,k\ge 3l} |2^{k/p}\mathcal{A}^{k,l} f_k|^2\Big)^{1/2}\Big\|_p \lesssim 2^{-\epsilon_1(p)l}\Big\|\Big(\sum_{k>0}|f_k|^2\Big)^{1/2}\Big\|_p. \tag{3.30}$$

where $\epsilon(p) > 0$, $p > (p_W + 2)/2$.

To verify (3.30) we follow closely an argument in [18] and use a vector-valued version of the Fefferman-Stein inequality for the #-function and linearization. The result in [18] does not apply but the method does if we replace certain estimates for singular integrals by $L^\infty \to BMO$ estimates for averaging operators (*cf.* the bound for (3.37) below).

Let us consider a family of cubes Q_x with $x \in Q_x$ so that the corners of Q_x are measurable functions, and suppose that

$$\sup_{x,y}\Big(\sum_k |g_k(x,y)|^2\Big)^{1/2} \le 1. \tag{3.31}$$

We define, for $k \ge 3l$, the linearized operator

$$T^z_{l,k} f(x) = 2^{k(1-z)/2} \frac{1}{|Q_x|} \int \Big[\mathcal{A}^{k,l} f(y) - \int \mathcal{A}^{k,l} f(u) \frac{du}{|Q_x|}\Big] g_k(x,y)\,dy, \tag{3.32}$$

and also

$$T^z_l F(x) = \sum_{k \ge 3l} T^z_{l,k} f_k(x) \quad \text{ where } F = \{f_k\} \in L^p(\ell^2).$$

The exponent $p(z)$ is given by $1/p(z) = (1-\mathrm{Re}(z))/2$. We shall have to prove that for $p(z) > (p_W+2)/2$ the operator T^z_l maps $L^{p(z)}(\ell^2)$ to $L^{p(z)}$ with operator norm $O(2^{-l\epsilon})$, independent of the choice of $Q(\cdot)$ and $g_k(\cdot,\cdot)$.

Split $T^z_l F = I^z_l F + II^z_l F + III^z_l F$, where

$$I^z_l F(x) = \sum_{\substack{k>0 \\ 2^{-10l} \le 2^k \mathrm{diam} Q_x \le 2^{10l}}} T^z_{l,k} f_k(x), \tag{3.33}$$

$$II^z_l F(x) = \sum_{\substack{k>0 \\ 2^k \mathrm{diam} Q_x \ge 2^{10l}}} T^z_{l,k} f_k(x), \tag{3.34}$$

$$III^z_l F(x) = \sum_{\substack{k>0 \\ 2^k \mathrm{diam} Q_x \le 2^{-10l}}} T^z_{l,k} f_k(x). \tag{3.35}$$

The main term is $I^z_l F(x)$ which is bounded by

$$\frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{\substack{k>0 \\ 2^{-10l} \le 2^k \mathrm{diam} Q_x \le 2^{10l}}} 2^{2k/p}\big|\mathcal{A}^{k,l}_t f_k(y) - \frac{1}{|Q_x|}\int_{Q_x} \mathcal{A}^{k,l}_t f_k(u)\frac{du}{|Q_x|}\big|^2\Big)^{1/2} dy$$
$$\lesssim (1+l)^{1/2-1/p}\Big(\sum_{k>0}\big[2^{k/p} M_{HL}(\mathcal{A}^{k,l} f_k)\big]^p\Big)^{1/p}(x),$$

by Hölder's inequality. Here $p = p(z)$ and M_{HL} denotes the Hardy-Littlewood maximal function. Now for $p = p(z) > (p_W+2)/2$,

$$\begin{aligned}\|I^z_l F\|_p &\lesssim (1+l)^{1/2-1/p} \sup_k 2^{k/p} \|\mathcal{A}^{k,l}\|_{L^p \to L^p} \|F\|_{L^p(\ell^p)}, \\ &\le C_p (1+l)^{1/2-1/p} 2^{-l\varepsilon_0(p)} \|F\|_{L^p(\ell^2)},\end{aligned}$$

by Corollary 3.4.

For the operators II^z_l and III^z_l we prove $L^2(\ell^2) \to L^2$ boundedness for $z = i\tau$ and $L^\infty(\ell^2) \to L^\infty$ boundedness for $z = 1 + i\tau$, with bounds uniform in τ. The $L^p(\ell^p) \to L^p$ estimate for $(1-\mathrm{Re}(z))/2 = 1/p$ then follows by analytic interpolation.

Using orthogonality arguments we obtain

$$\|II^{i\tau}_l F\|_2 + \|III^{i\tau}_l F\|_2 \le C_\epsilon 2^{l(\frac12+\epsilon)} \|F\|_{L^2(\ell^2)} \tag{3.36}$$

for arbitrary $\epsilon > 0$. To see this, let us consider $II_l^{i\tau}$. By (3.31)

$$\begin{aligned}&\big|II_l^{i\tau}F(x)\big|\\&\le \sum_{2^k \operatorname{diam}(Q_x)\ge 2^{10l}} \frac{2^{k/2}}{|Q_x|}\int_{Q_x}\Big|\mathcal{A}^{k,l}f_k(y)-\frac{1}{|Q_x|}\int_{Q_x}\mathcal{A}^{k,l}f_k(u)\,du\Big||g_k(x,y)|\,dy\\&\le \Big(\sum_{k>0}\big|2^{\frac{k}{2}}\big(M_{HL}(\mathcal{A}^{k,l}f_k)\big)(x)\big|^2\Big)^{1/2}.\end{aligned}$$

Therefore,

$$\|II_l^{i\tau}F\|_2 \le \Big(\sum_k \|2^{\frac{k}{2}}\mathcal{A}^{k,l}f_k\|_2^2\Big)^{1/2} \le C2^{l/2}\|F\|_{L^2(\ell^2)},$$

where we have used (3.29). This proves (3.36) for $II_l^{i\tau}$ and the argument for $III_l^{i\tau}$ is exactly analogous.

For the L^∞ bounds let us consider $II_l^{1+i\tau}F(x)$ for fixed x and $\mathrm{Re}(z)=1$. We note that

$$II_l^{1+i\tau}F(x) \le 2\frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{2^k\operatorname{diam}(Q_x)>2^{10l}}\big|\mathcal{A}^{k,l}f_k(y)\big|^2\Big)^{1/2}dy. \tag{3.37}$$

Let

$$\mathcal{U}(x)=\{y: |x-y+\gamma(s)|\le 10\operatorname{diam}(Q_x),\ \text{for some } s\in\operatorname{supp}(\chi).\}$$

Then

$$|\mathcal{U}(x)|\lesssim \big(\operatorname{diam}(Q_x)\big)^2.$$

We estimate

$$II_l^{1+i\tau}F(x)\le 2\big[II_{l,1}^{1+i\tau}F(x)+II_{l,2}^{1+i\tau}F(x)\big]$$

where

$$\begin{aligned}II_{l,1}^{1+i\tau}F(x)&=\frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{2^k\operatorname{diam}(Q_x)>2^{10l}}\big|\mathcal{A}^{k,l}[\chi_{\mathcal{U}(x)}f_k](y)\big|^2\Big)^{1/2}dy,\\II_{l,2}^{1+i\tau}F(x)&=\frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{2^k\operatorname{diam}(Q_x)>2^{10l}}\big|\mathcal{A}^{k,l}[\chi_{\mathbb{R}^3\setminus\mathcal{U}(x)}f_k](y)\big|^2\Big)^{1/2}dy.\end{aligned}$$

The term $II_{l,1}^{1+i\tau}F(x)$ is estimated by an L^2 estimate; we obtain after applying the Cauchy-Schwarz inequality, (almost) orthogonality of the $\mathcal{A}^{k,l}$ and (3.29),

$$\begin{aligned}|II_{l,1}^{1+i\tau}F(x)|&\le\Big(\frac{1}{|Q_x|}\int\sum_{2^k\operatorname{diam}(Q_x)>2^{10l}}\big|\mathcal{A}^{k,l}[\chi_{\mathcal{U}(x)}f_k](y)\big|^2dy\Big)^{1/2},\\&\lesssim \sup_{2^k\operatorname{diam}(Q_x)>2^{10l}}\|\mathcal{A}^{k,l}\|_{L^2\to L^2}\Big(\frac{1}{|Q_x|}\Big\|\Big(\sum_k|\chi_{\mathcal{U}(x)}f_k|^2\Big)^{1/2}\Big\|_2^2\Big)^{1/2}\\&\le C_\epsilon\sup_{2^k\operatorname{diam}(Q_x)>2^{10l}}2^{(l-k)/2+l\epsilon}\Big(\frac{|\mathcal{U}(x)|}{|Q_x|}\Big)^{1/2}\|F\|_{L^\infty(\ell^2)}\\&\le C_\epsilon 2^{-9l/2+l\epsilon}\|F\|_{L^\infty(\ell^2)},\end{aligned}$$

where at the last step we have used the fact that $|\mathcal{U}(x)|/|Q_x|\lesssim\operatorname{diam}(Q_x)^{-1}$.

We now crudely estimate the terms $II^z_{l,2}F(x)$ and $III^z_l F(x)$, $z = 1 + i\tau$. For this we make use of the following pointwise estimate obtained from integration by parts :

$$(3.38)\quad |\mathcal{A}^{k,l} f(y)| + 2^{-k}|\nabla \mathcal{A}^{k,l} f(y)| \lesssim \iint \frac{2^{3k-2l}}{(1+2^{k-2l}|y-w+\gamma(s)|)^N} |f(w)| dw\, ds,$$

with $N \ge 4$. To estimate $|II^{1+i\tau}_{l,2}F(x)|$ we need (3.38) for $y \in Q_x$ and $w \notin \mathcal{U}(x)$, i.e. $|y - w + \gamma(s)| \gtrsim \mathrm{diam}(Q_x)$ for all relevant s. This yields the bound

$$|T^{1+i\tau}_{l,k} f_k(x)| \lesssim \int_s \frac{1}{|Q_x|} \int_{Q_x} \int_{\substack{|y-z+\gamma(s)| \\ \ge c\mathrm{diam}(Q_x)}} \frac{2^{3k-2l}2^{(2l-k)N}}{|y-z+\gamma(s)|^N} |f_k(z)| dz\, dy\, ds$$
$$\lesssim 2^{3k-2l}2^{(2l-k)N}(\mathrm{diam}(Q_x))^{3-N}\|F\|_{\ell^\infty(L^\infty)}.$$

Therefore,

$$\sum_{2^k \mathrm{diam}(Q_x) > 2^{10l}} |T^{1+i\tau}_{l,k} f_k(x)| \lesssim 2^{2l(N-1)}2^{10l(3-N)}\|F\|_{\ell^\infty(L^\infty)},$$

which certainly implies

$$(3.39)\qquad |II^{1+i\tau}_{l,2}F(x)| \lesssim 2^{-4l}\|F\|_{L^\infty(\ell^2)}.$$

To estimate $III^z_l F(x)$ we use instead the estimate for the gradient in (3.38) and get

$$|T^{1+i\tau}_{l,k} f_k(x)| \le \int_{Q_x}\int_{Q_x} \Big| \int_{\sigma=0}^1 \langle y - z, \nabla \mathcal{A}^{k,l} f(y + \sigma(z-y))\rangle d\sigma \Big| \frac{dz}{|Q_x|}\frac{dy}{|Q_x|}$$
$$\le 2^{4l}2^k \mathrm{diam}(Q_x)\|f_k\|_\infty.$$

We sum over k with $2^k\mathrm{diam}(Q_x) \le 2^{-10l}$ and obtain

$$(3.40)\qquad |III^{1+i\tau}_l F(x)| \lesssim 2^{-6l}\|F\|_{\ell^\infty(L^\infty)} \lesssim 2^{-6l}\|F\|_{L^\infty(\ell^2)}.$$

Interpolating the bounds (3.39) and (3.40) with (3.36) we obtain (3.30) with $\epsilon(p) > 0$ for a range of p's which includes $(4, \infty)$ and therefore $((p_W + 2)/2, \infty)$.

We observe that by choosing a parameter larger than 10 in the definition of I, II, III we could enlarge the range where $\epsilon(p) > 0$ in (3.30), but this is irrelevant here. This finishes the proof of Theorem 1.1 in the case of nonvanishing curvature and torsion. □

Extension to finite type curves.

We now consider the averaging operator $\mathcal{A}_t$ as in (1.3) and assume that γ is of maximal type n. We shall fix s_0 and estimate $\mathcal{A}_t$ under the assumption that the cutoff function χ is supported in a small neighborhood of s_0. This assumption implies that there are orthogonal unit vectors $\theta_1, \theta_2, \theta_3$ and integers $1 \le n_1 < n_2 < n_3 \le n$ so that for $i = 1, 2, 3$,

$$\langle \theta_i, \gamma^{(j)}(s_0)\rangle = 0, \text{ if } 1 \le j < n_i, \quad \langle \theta_i, \gamma^{(n_i)}(s_0)\rangle \ne 0.$$

After a rotation we may also assume that $\theta_i = e_i$, $i = 1, 2, 3$, and

$$(3.41)\qquad \gamma(s_0 + \alpha) = \gamma(s_0) + (\beta_1\alpha^{n_1}\varphi_1(\alpha), \beta_2\alpha^{n_2}\varphi_2(\alpha), \beta_3\alpha^{n_3}\varphi_3(\alpha))$$

where β_1, β_2, β_3 are nonzero constants and $\varphi_i \in C^{n+5-n_i}$ with $\varphi_i(0) = 1$. Thus we need to establish the asserted $L^p \to L^p_{1/p}$-boundedness for the averages

$$\mathcal{A}_t f(x) = \int \chi(\alpha) f(x - t\gamma(s_0 + \alpha)) d\alpha$$

with bounds uniformly in $t \in [1/2, 2]$, where χ is chosen so that we assume that $1/2 \le |\varphi_i(\alpha)| \le 3/2$, $i = 1, 2, 3$, in the support of χ. We work with a dyadic partition of unity ζ_j, where $\zeta_j = \zeta(2^j \cdot)$ is supported where $|\alpha| \approx 2^{-j}$; we also set $\chi_j = \chi\zeta_j(2^{-j})$ so that the derivatives of χ_j are bounded independently of j. Let

$$\begin{aligned} A_{j,t} f(x) :&= \int \chi_j(2^j\alpha) f(x - t\gamma(s_0 + \alpha)) d\alpha \\ &= 2^{-j} \int \chi_j(u) f(x - t\gamma(s_0 + 2^{-j}u)) du \end{aligned} \tag{3.42}$$

so that $\mathcal{A}_t = \sum_{j>0} A_{j,t}$. Now set

$$\begin{aligned} \delta_j(x) &= (2^{jn_1}x_1, 2^{jn_2}x_2, 2^{jn_3}x_3) \\ \Gamma_j(u) &= \big(\beta_1 u^{n_1}\varphi_1(2^{-j}u), \beta_2 u^{n_2}\varphi_2(2^{-j}u), \beta_3 u^{n_3}\varphi_3(2^{-j}u)\big) \end{aligned} \tag{3.43}$$

A change of variable shows that

$$\begin{aligned} A_{j,t} f(x) &= \\ & 2^{-j} \int f_{-j}(\delta_j x - t\delta_j\gamma(s_0) - t\Gamma_j(u))\chi_j(u) du, \quad \text{where } f_{-j}(y) = f(\delta_{-j}y). \end{aligned} \tag{3.44}$$

We note that the curves Γ_j have C^{n+5-n_j} bounds (in particular C^5 bounds) independent of j, and that the parameter u belongs to the union of two intervals $\pm(c_1, c_2)$ away from the origin (with c_1, c_2 independent of j). Moreover

$$\det \big(\Gamma_j'(u) \quad \Gamma_j''(u) \quad \Gamma_j'''(u)\big) \approx \beta_1\beta_2\beta_3 u^{n_1+n_2+n_3}(1 + O(2^{-j}))$$

so that the uniform results in the case of nonvanishing curvature and torsion apply. Observe that for $\alpha \ge 0$

$$\|g \circ \delta_j\|_{L^p_\alpha} \lesssim 2^{j(n_3\alpha - N/p)} \|g\|_{L^p_\alpha}, \qquad N = n_1 + n_2 + n_3.$$

Thus for $p > (p_W + 2)/2$,

$$\begin{aligned} \big\|A_{j,t} f\big\|_{L^p_{1/p}} &\lesssim 2^{-j} 2^{j(n_3/p - N/p)} \Big\| \int f_{-j}(\cdot - t\delta_j\gamma(s_0) - t\Gamma_j(u))\chi_j(u) du \Big\|_{L^p_{1/p}} \\ &\lesssim 2^{j(-1+n_3/p-N/p)} \|f \circ \delta_{-j}\|_p = 2^{j(-1+n_3/p)} \|f\|_p. \end{aligned}$$

Since we assume that $p > n \ge n_3$ we can sum in j to arrive at the desired conclusion. □

4. Microlocal Smoothing Estimates for Curves in $\mathbb{R}^d$

In this section we consider a C^3 curve $u \mapsto \Gamma(u)$ in $\mathbb{R}^d$, defined in a compact interval J, and we assume that there is a constant $B \ge 1$ so that $B^{-1} \le |J| \le B$ and for all $u \in J$

$$\sum_{i=1}^{3} |\Gamma^{(i)}(u)| \le B. \tag{4.1}$$

We study the space-time smoothing properties of the averaging operator, when localized to the region where $|\langle\Gamma''(u),\xi\rangle| \approx |\xi|$. Consider for a compactly supported symbol a the operator defined by

$$\mathfrak{A}_\Gamma[a,f](x,t) = (2\pi)^{-d} \iint a(u,t,\xi) e^{i\langle x,\xi\rangle - it\langle\Gamma(u),\xi\rangle} \widehat{f}(\xi)\, du\, d\xi. \tag{4.2}$$

Theorem 4.1. *Let J_0 be the closed subinterval of J with same center and length $|J|/2$, and assume that a is supported in*

$$J_0 \times [1/2,2] \times \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$$

and that the inequalities

$$|\partial_u^{i_1}\partial_t^{i_2}\partial_\xi^\alpha a(u,t,\xi)| \le \mathcal{C}[a]|\xi|^{-|\alpha|}, \tag{4.3}$$

hold for $|\alpha| \le d+2$, $0 \le i_1 \le 1$, $0 \le i_2 \le 1$. Moreover assume that

$$\big|\langle\Gamma'(u),\xi\rangle\big| + \big|\langle\Gamma''(u),\xi\rangle\big| \ge B^{-1}|\xi| \quad \text{if } (u,t,\xi) \in \mathrm{supp}(a). \tag{4.4}$$

Then for $p > p_W$ and $f \in L^p(\mathbb{R}^d)$

$$\big\|\mathfrak{A}_\Gamma[a,f]\big\|_{L^p(\mathbb{R}^{d+1})} \le C(\varepsilon,p,B,d)\,\mathcal{C}[a]\,2^{-k(\frac{2}{p}-\varepsilon)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{4.5}$$

The crucial hypothesis on Γ is the lower bound (4.4). We note that the derivatives of Γ are assumed to be bounded but we make no size assumption on $|\Gamma(u)|$ itself. Thus the assumptions on Γ are invariant under translation of the curve.

In the following subsection we shall prove this theorem under slightly more restrictive normalization assumptions which will be removed at the end of this section by localization and scaling arguments.

4.1. **Normalization.** We now work with a C^3 curve $s \mapsto \gamma(s)$, $\gamma : I^* \to \mathbb{R}^d$, $d \ge 3$, where $I^* = [-2\delta, 2\delta]$ is a closed subinterval of $[-1,1]$; we also set $I = [-\delta,\delta]$. We assume that the curve is parametrized by arclength, i.e. $|\gamma'(s)| = 1$ for all $s \in I^*$ and that for some $M \ge 10$,

$$|\gamma''(s)| + |\gamma'''(s)| \le M. \tag{4.6}$$

Let Ω be an open convex conic subset of $\mathbb{R}^d \setminus \{0\}$, and let

$$\Omega_k = \{\xi \in \Omega : 2^{k-1} \le |\xi| \le 2^{k+1}\}. \tag{4.7}$$

We shall study $\mathfrak{A}_\gamma[b,f]$ defined as in (4.2) and we now assume that the symbol b is supported in $I \times [1/2,2] \times \Omega_k$ and satisfies

$$|\partial_s^{i_1}\partial_t^{i_2}\partial_\xi^\alpha b(s,t,\xi)| \le C_{i_1,i_2,\alpha}|\xi|^{-|\alpha|}, \tag{4.8}$$

for $|\alpha| \le d+2$, $0 \le i_1 \le 1$, $0 \le i_2 \le 1$. We assume that Ω satisfies the crucial

Nondegeneracy Hypothesis:

$$|\xi|/2 \le |\langle\gamma''(s),\xi\rangle| \le 2|\xi| \tag{4.9}$$

for all $\xi \in \Omega$, $s \in I$; moreover we assume that for every ξ in Ω there is at least one $s \in [-3\delta/4, 3\delta/4]$ so that

$$|\langle\gamma'(s),\xi\rangle| \le \delta|\xi|/10. \tag{4.10}$$

Note that the smallness assumption (4.10) and the lower bound (4.9) imply that for each $\xi \in \Omega$ there is a unique $s = s_{\mathrm{cr}}(\xi)$ in $(-\delta,\delta)$ so that

$$\langle\gamma'(s),\xi\rangle = 0 \iff s = s_{\mathrm{cr}}(\xi), \tag{4.11}$$

and $\xi \mapsto s_{\rm cr}(\xi)$ is a C^3 function on Ω which is homogeneous of degree 0.

The next subsection is devoted to the proof of

Theorem 4.2. *Assume that $k \ge 10$, $\varepsilon_0 > 0$, that b is supported in $I \times [1/2, 2] \times \Omega_k$ and that (4.6) (4.8), and the nondegeneracy hypothesis hold.*

Then for $p > p_W$

$$\big\|\mathfrak{A}_\gamma[b, f]\big\|_{L^p(\mathbb{R}^{d+1})} \le C(\varepsilon_0, p, M, d) 2^{-k(\frac{2}{p}-\varepsilon_0)} \|f\|_{L^p(\mathbb{R}^d)}, \quad \varepsilon_0 > 0. \tag{4.12}$$

4.2. **Proof of Theorem 4.2.** We use the following

Notation: "Constants" C may depend on M and the dimension; we shall use the Landau symbol $\mathcal{E} = O(B)$ if $|\mathcal{E}| \le CB$. We shall also use the notation $\mathcal{E} = \mathcal{O}_1(B)$ if $|\mathcal{E}| \le B$.

Some symbol classes. Let $2^{-k/2} < r \le \min\{10^{-3}M^{-2}, \delta/4\}$, $s \in (-3\delta/4, 3\delta/4)$. We define some symbol classes for multipliers $m(\xi, \tau)$ and set $\Xi = (\xi, \tau)$, with $\tau \equiv \Xi_{d+1}$. We denote by e_1, ..., e_{d+1} the standard basis in $\mathbb{R}^{d+1}$. Let $L_s^{(1)}$ be the linear shear transformation which maps

$$\Xi = \sum_{i=1}^d \xi_i e_i + \tau e_{d+1} \ \mapsto\ L_s^{(1)}\Xi = \sum_{i=1}^d \xi_i e_i + (\tau - \langle\gamma(s), \xi\rangle) e_{d+1}.$$

Let $L_{r,s}^{(2)}$ be the dilation which satisfies

$$L_{r,s}^{(2)} e_{d+1} = r^2 e_{d+1}, \quad L_{r,s}^{(2)}\gamma'(s) = r\gamma'(s)$$
$$L_{r,s}^{(2)} v = v \text{ if } v \in \big(\operatorname{span}\{e_{d+1}, \gamma'(s)\}\big)^\perp;$$

here we identify with a slight abuse of notation the function γ with the function $s \to (\gamma(s), 0)$ with values in $\mathbb{R}^{d+1}$. We define the composition

$$L_{r,s} = L_s^{(1)} L_{r,s}^{(2)}.$$

Let $\mathcal{S}_k(r, s)$ be the class of multipliers $m(\xi, \tau)$ which are supported in

$$\big\{\Xi = (\xi, \tau) : 2^{k-1} \le |\xi| \le 2^{k+1}, \\ |\langle\gamma'(s), \xi\rangle| \le 2^{k+3} r, \quad |\tau + \langle\gamma(s), \xi\rangle| \le 2^{k+4} r^2\big\} \tag{4.13}$$

and satisfy

$$\big|\partial_\Xi^\alpha\big(m(L_{r,s}\Xi)\big)\big| \le |\Xi|^{-\alpha}, \quad |\alpha| \le d+2. \tag{4.14}$$

Note that if $m \in \mathcal{S}_k(r, s)$ and $(\xi, \tau) \in \operatorname{supp}(m)$ then $|L_{r,s}^{-1}\Xi| = O(2^k)$. The following Lemma is straightforward to check, we omit the proof.

Lemma 4.2.1. *There are constants $C_i = C_i(A, M)$, $i = 1, 2$, so that $C_1^{-1} m \in \mathcal{S}_k(C_2 r, s')$ for all $m \in \mathcal{S}_k(r, s)$ and all s' with $|s - s'| \le Ar$.*

We shall need kernel estimates for operators associated with multipliers in $\mathcal{S}_k(r, s)$.

Lemma 4.2.2. *Let $m(t, \cdot) \in \mathcal{S}_k(r, s')$ for $1/2 \le t \le 2$ (depending continuously on t) and assume $|s' - s| \le 2r$. Let*

$$K_s[m](x, t') = \iiint e^{i\langle x,\xi\rangle + it'\tau} e^{-it(\tau + \langle\gamma(s),\xi\rangle)} m(t, \xi, \tau)\, d\xi d\tau\, dt. \tag{4.15}$$

Then

$$|K_s[m](x,t')| \le C \times \int \frac{2^{k(d+1)}r^3}{(1+2^kr^2|t-t'|+2^kr|\langle x-t'\gamma(s),\gamma'(s)\rangle|+2^k|\Pi^{\perp}_{\gamma'(s)}(x-t'\gamma(s))|)^{d+2}}\,dt \tag{4.16}$$

where $\Pi^{\perp}_{\gamma'(s)}:\mathbb{R}^d\to\mathbb{R}^d$ *denotes the orthogonal projection to the orthogonal complement of* $\mathbb{R}\gamma'(s)$*. In particular*

$$\sup_{x,t'}\int |K_s[m](x-y,t')|dy \le C.$$

Proof. The second assertion is an immediate consequence of (4.16). To see (4.16) we change variables in the integral defining K_s and see that

$$K_s[m](x,t') = \int e^{i(t'-t)\tau}e^{i\langle x-t'\gamma(s),\xi\rangle}m(t,L_s^{(1)}\Xi)d\Xi dt \tag{4.17}$$

with $\Xi=(\xi,\tau)$. Changing variables again using $\Xi=L^{(2)}_{r,s}\widetilde{\Xi}$ and integrating by parts in $\widetilde{\Xi}$ yields (4.16). □

Now let $s_0\in(-3\delta/4,3\delta/4)$, $r\le\delta/8$ and let $\mathfrak{S}_k(r,s_0)$ be the class of symbols $(s,t,\xi,\tau)\mapsto a(s,t,\xi,\tau)$ which are supported in $[s_0-2r,s_0+2r]\times[1/2,2]\times\Omega_k\times\mathbb{R}$ and which satisfy

$$\partial_s^{i_1}\partial_t^{i_2}a(s,t,\cdot)\in\mathcal{S}_k(r,s_0),\qquad \text{for } i_1,i_2\in\{0,1\}.$$

We define the oscillatory integral

$$\mathcal{T}[a,f](x,t)=(2\pi)^{-d-1}\iint e^{i(\langle x,\xi\rangle+t\tau)}\mathfrak{m}[a](\xi,\tau)\widehat{f}(\xi)d\xi d\tau \tag{4.18}$$

where

$$\mathfrak{m}[a](\xi,\tau)=\iint e^{-it(\tau+\langle\gamma(s),\xi\rangle)}a(s,t,\xi,\tau)dsdt. \tag{4.19}$$

The proof of Theorem 4.2 relies on an iteration where the main step is to prove the following proposition. Here we say that a set of real numbers is *r-separated* if $|s-s'|\ge r$ for different s, s' in this set.

Proposition 4.2.3. *Suppose that* $\varepsilon>0$, $r_1\le r_0\le\min\{10^{-3}M^{-2},\delta/4\}$*, and assume that* $r_1\ge 100Mr_0^{3/2}$*. Let* $\{s^\mu\}$ *be an* r_0*-separated set of points in* $[-\delta,\delta]$*, and for each* μ *let* a^μ *be a symbol in* $\mathfrak{S}_k(r_0,s^\mu)$*. Let* $p>p_W$*. Then there is a set of* r_1*-separated points* $\{s_\nu\}$ *and symbols* $a_\nu\in\mathfrak{S}_k(r_1,s_\nu)$*, and for every* $\varepsilon>0$ *there is a constant* $\mathcal{C}_\varepsilon=\mathcal{C}_\varepsilon(p,M)$*, so that*

$$\Big(\sum_\mu \big\|\mathcal{T}[a^\mu,f]\big\|_p^p\Big)^{1/p} \le \mathcal{C}_\varepsilon(r_0/r_1)^{1-\frac4p+\varepsilon}\Big[\Big(\sum_\nu\big\|\mathcal{T}[a_\nu,f]\big\|_p^p\Big)^{1/p}+2^{-k}r_1^{-1}\|f\|_p\Big] \tag{4.20}$$

holds.

Proof of the Proposition.

For each μ we set

$$U_\mu(\xi,\tau) = \tau + \langle \gamma(s^\mu),\xi\rangle - \frac{1}{2}\frac{\langle \gamma'(s^\mu),\xi\rangle^2}{\langle \gamma''(s^\mu),\xi\rangle}. \tag{4.21}$$

The second part of the following lemma states that U_μ is a good approximation for $\tau + \langle \gamma(s_{\mathrm{cr}}(\xi)),\xi\rangle$.

Lemma 4.2.4. *Suppose $|s| \le 3\delta/4$, $\xi\in\Omega$. Then*

$$s - s_{cr}(\xi) = \frac{\langle \gamma'(s),\xi\rangle}{\langle \gamma''(s),\xi\rangle} + \mathcal{O}_1(6M(s-s_{cr}(\xi))^2); \tag{4.22}$$

in particular this holds for $s=s^\mu$ if $(\xi,\tau)\in \mathrm{supp}(a^\mu)$ for some τ. Moreover

$$U_\mu(\xi,\tau) \;=\; \tau + \langle\gamma(s_{cr}(\xi)),\xi\rangle + \mathcal{O}_1(13M|s_{cr}(\xi)-s^\mu|^3)|\xi| \tag{4.23}$$

if $(\xi,\tau)\in\mathrm{supp}(a^\mu)$.

Proof. We expand using (4.11)

$$\begin{aligned}\langle \gamma'(s),\xi\rangle &= \langle \gamma''(s_{\mathrm{cr}}(\xi)),\xi\rangle(s-s_{\mathrm{cr}}(\xi)) + \mathcal{O}_1(M|\xi|(s-s_{\mathrm{cr}}(\xi))^2/2)\\ &= \langle \gamma''(s),\xi\rangle(s-s_{\mathrm{cr}}(\xi)) + \mathcal{O}_1(3M|\xi|(s-s_{\mathrm{cr}}(\xi))^2/2)\end{aligned} \tag{4.24}$$

and (4.22) follows by using the lower bound in (4.9).

Next expand again using (4.11)

$$\begin{aligned}&\langle\gamma(s^\mu),\xi\rangle - \langle\gamma(s_{\mathrm{cr}}(\xi)),\xi\rangle\\ &\quad=\langle\gamma''(s_{\mathrm{cr}}(\xi)),\xi\rangle\frac{(s^\mu-s_{\mathrm{cr}}(\xi))^2}{2} + \mathcal{O}_1(M|\xi||s^\mu-s_{\mathrm{cr}}(\xi)|^3/6)\\ &\quad=\langle\gamma''(s^\mu),\xi\rangle\frac{(s^\mu-s_{\mathrm{cr}}(\xi))^2}{2} + \mathcal{O}_1(2M|\xi||s^\mu-s_{\mathrm{cr}}(\xi)|^3/3).\end{aligned}$$

Now we use (4.22) for $s=s^\mu$ and get

$$\begin{aligned}&\langle\gamma''(s^\mu),\xi\rangle\frac{(s^\mu-s_{\mathrm{cr}}(\xi))^2}{2} - \frac{\langle\gamma'(s^\mu),\xi\rangle^2}{2\langle\gamma''(s^\mu),\xi\rangle}\\ &= \langle\gamma'(s^\mu),\xi\rangle\mathcal{O}_1(6M(s^\mu-s_{\mathrm{cr}}(\xi))^2) + \mathcal{O}_1(18M^2(s^\mu-s_{\mathrm{cr}})^4)|\xi|\\ &= \mathcal{O}_1(12M(s^\mu-s_{\mathrm{cr}}(\xi))^3)|\xi| + \mathcal{O}_1(20M^2(s^\mu-s_{\mathrm{cr}})^4)|\xi|.\end{aligned}$$

Since we assume that $r\le 10^{-3}M^{-1}$ we obtain (4.23). □

We now decompose a^μ using cutoff functions η_0, η_1, ζ as in §3, that is, η_0 is supported in $[-1,1]$, equal to 1 in $[-1/2,1/2]$, $\eta_1=\eta_0(4^{-1}\cdot)-\eta_0$, and ζ is supported in $(-1,1)$ and satisfies $\sum_\nu\zeta(s-\nu)=1$, $s\in\mathbb{R}$. These cutoff functions are fixed and various constants below may depend on their choice. Set

$$a^\mu_{0,\nu}(s,t,\xi,\tau) = a^\mu(s,t,\xi,\tau)\eta_0\big(r_1^{-2}(2^{-k}|U_\mu(\xi,\tau)| + (s-s_{\mathrm{cr}}(\xi))^2)\big)\zeta(r_1^{-1}s-\nu),$$

and, for $n\ge 1$

$$\begin{aligned} a^\mu_{n,\nu}(s,t,\xi,\tau) =& a^\mu(s,t,\xi,\tau)\eta_1\big(2^{2-2n}r_1^{-2}(2^{-k}|U_\mu(\xi,\tau)|+(s-s_{\rm cr}(\xi))^2)\big)\\ &\times\ \eta_0\big(\frac{(s-s_{\rm cr}(\xi))^2}{2^{-k-8}U_\mu(\xi,\tau)}\big)\zeta(2^{-n}r_1^{-1}s-\nu),\\ b^\mu_{n,\nu}(s,t,\xi,\tau) =& a^\mu(s,t,\xi,\tau)\,\eta_1\big(2^{2-2n}r_1^{-2}(2^{-k}|U_\mu(\xi,\tau)|+(s-s_{\rm cr}(\xi))^2)\big)\\ &\times\ \big(1-\eta_0\big(\frac{(s-s_{\rm cr}(\xi))^2}{2^{-k-8}U_\mu(\xi,\tau)}\big)\big)\zeta(2^{-n}r_1^{-1}s-\nu)\end{aligned}$$

Then

$$a^\mu=\sum_\nu a^\mu_{0,\nu}+\sum_{n\ge1}\sum_\nu(a^\mu_{n,\nu}+b^\mu_{n,\nu}). \tag{4.25}$$

In what follows we define the linear map $\omega^\mu:\mathbb{R}^{d+1}\to\mathbb{R}^3$ by

$$\begin{cases}\omega^\mu_1(\xi,\tau) &=\langle\gamma'(s^\mu),\xi\rangle,\\ \omega^\mu_2(\xi,\tau) &=\tau+\langle\gamma(s^\mu),\xi\rangle,\\ \omega^\mu_3(\xi,\tau) &=\langle\gamma''(s^\mu),\xi\rangle.\end{cases} \tag{4.26}$$

We shall observe that for fixed μ the supports of $a^\mu_{n,\nu}$ and $b^\mu_{n,\nu}$ are contained in "plates" defined using $\omega^\mu(\xi,\tau)$ (*cf.* (4.27) below).

Lemma 4.2.5. *(i) Suppose (ξ,τ) is in the support of $a^\mu_{n\nu}(s,t,\cdot)$ or $b^\mu_{n\nu}(s,t,\cdot)$ then $|U_\mu(\xi,\tau)|\le 2^{k+2n}r_1^2$, and $|s-s_{cr}(\xi)|\le 2^nr_1$.*

(ii) Suppose $n\ge1$.

If (ξ,τ) is in the support of $a^\mu_{n\nu}(s,t,\cdot)$ then $|U_\mu(\xi,\tau)|\ge 2^{k+2n-3}r_1^2$.

If (ξ,τ) is in the support of $b^\mu_{n\nu}(s,t,\cdot)$ then $|s-s_{cr}(\xi)|\ge 2^{n-5}r_1$.

(iii) Let $s_{n\nu}=2^nr_1\nu$ for $\nu\in\mathbb{Z}$ and assume that $|s_{n\nu}-s^\mu|\le 2r_0$. Then there is a constant C so that the symbols $C^{-1}a^\mu_{n,\nu}$, $C^{-1}b^\mu_{n,\nu}$ belong to $\mathfrak{S}_k(2^nr_1,s_{n\nu})$.

(iv) If $2^nr_1>2^4r_0$ then $a^\nu_{n,\nu}=0$.

(v) If $2^nr_1>2^7r_0$ then $b^\nu_{n,\nu}=0$.

(vi) Let $g(\alpha)=(\alpha,\alpha^2/2)$ and let $u_i(\alpha)$, $i=1,2,3$, be as in (2.2), i.e.

$$u_1(\alpha)=(\alpha,\alpha^2/2,1),\ u_2(\alpha)=(1,\alpha,0),\ u_3(\alpha)=(-\alpha,1,\alpha^2/2).$$

Let $\overline{\alpha}=\overline{\alpha}_{\mu n\nu}=s^\mu-s_{n\nu}$. Then the supports of $a^\mu_{n,\nu}$ and $b^\mu_{n,\nu}$ are contained in the set $Pl^n_{\mu\nu}$ consisting of all (ξ,τ) which satisfy

$$\begin{cases}|\langle u_1(\overline{\alpha}),\omega^\mu(\xi,\tau)\rangle|\le 2^{k+2},\\ |\langle u_2(\overline{\alpha}),\omega^\mu(\xi,\tau)-\omega^\mu_3(\xi,\tau)u_1(\overline{\alpha})\rangle|\le 2^{k+4}2^nr_1,\\ |\langle u_3(\overline{\alpha}),\omega^\mu(\xi,\tau)\rangle|\le 2^{k+3}2^{2n}r_1^2.\end{cases} \tag{4.27}$$

(vii) Every (ξ,τ) belongs to no more than 75 of the sets $\{(\xi,\tau):(s,t,\xi,\tau)\in\operatorname{supp}(a^\mu_{n,\nu})\}$ and to no more than 75 of the sets $\{(\xi,\tau):(s,t,\xi,\tau)\in\operatorname{supp}(b^\mu_{n,\nu})\}$.

Proof. Properties (i) and (ii) are immediate consequences of the definition of the symbols.

For (iii) we first have to check the support properties, namely assuming that (s,t,ξ,τ) belongs to the support of $a^\mu_{n,\nu}$ or $b^\mu_{n,\nu}$ then

$$|\langle\gamma'(s_{n\nu}),\xi\rangle|\le 2^{k+3}2^nr_1 \tag{4.28}$$

$$|\tau+\langle\gamma(s_{n\nu}),\xi\rangle|\le 2^{k+4}2^{2n}r_1^2 \tag{4.29}$$

To see this we first note that $|s-s_{n\nu}|\le 2^n r_1$ and $(s-s_{\rm cr}(\xi))^2\le 2^{2n}r_1^2$ hence

$$|s_{\rm cr}(\xi)-s_{n\nu}|\le 2^{n+1}r_1 \tag{4.30}$$

Similarly we have of course also

$$|s_{\rm cr}(\xi)-s^\mu|\le 2r_0 \tag{4.31}$$

Now $\langle\gamma'(s_{n\nu}),\xi\rangle=\langle\gamma''(\tilde s),\xi\rangle(s_{n\nu}-s_{\rm cr}(\xi))$ where $\tilde s$ is between $s_{n\nu}$ and $s_{\rm cr}(\xi)$. Since $|\langle\gamma''(\tilde s),\xi\rangle|\le 2^{k+2}$ we conclude (4.28).

To see (4.29) we expand

$$\tau+\langle\gamma(s_{n\nu}),\xi\rangle=\tau+\langle\gamma(s_{\rm cr}(\xi)),\xi\rangle+\langle\gamma''(s'),\xi\rangle\frac{(s_{n\nu}-s_{\rm cr}(\xi))}{2}$$

where s' is between $s_{n\nu}$ and $s_{\rm cr}(\xi)$. From (4.23) and (4.30) we obtain

$$|\tau+\langle\gamma(s_{n\nu}),\xi\rangle|\le 2^{k+3}2^{2n}r_1^2+|U_\mu(\xi,\tau)|+13M(2r_0)^3|\xi|.$$

Now $|U_\mu(\xi,\tau)|\le 2^{k+2n}r_1^2$ and from our crucial assumption on the relation between r_0 and r_1, namely $r_0^3\le(100M)^{-2}r_1^2$, we can deduce (4.29).

We now have to verify the symbol estimates (4.14). First observe $\partial_\tau U_\mu=1$ and calculate (using the notation in (4.26))

$$\nabla_\xi U_\mu=\gamma(s^\mu)-\frac{\omega_1^\mu}{\omega_3^\mu}\gamma'(s^\mu)+\frac12\Big(\frac{\omega_1^\mu}{\omega_3^\mu}\Big)^2\gamma''(s^\mu)$$

and an expansion about the point $s_{n\nu}$ yields that

$$\begin{aligned}\nabla_\xi U_\mu=\gamma(s_{n\nu})&+\gamma'(s_{n\nu})\big(s^\mu-s_{n\nu}-\frac{\omega_1^\mu}{\omega_3^\mu}\big)\\&+\gamma''(s_{n\nu})\Big(\frac{(s^\mu-s_{n\nu})^2}{2}-\frac{\omega_1^\mu}{\omega_3^\mu}(s^\mu-s_{n\nu})+(\frac12\Big(\frac{\omega_1^\mu}{\omega_3^\mu}\Big)^2\Big)+O(r_0^3).\end{aligned}$$

A further expansion using (4.24) shows that on the support of either $a^\mu_{n,\nu}$ $(n\ge0)$ or $b^\mu_{n,\nu}$ $(n\ge1)$

$$\nabla_\xi U_\mu=\gamma(s_{n\nu})+\gamma'(s_{n\nu})O(2^nr_1)+O(2^{2n}r_1^2)+O(r_0^3).$$

Thus if v is perpendicular to $(\gamma(s_{n\nu}),1)$ then $\langle v,\nabla\rangle U_\mu=O(2^nr_1)$ and if v is perpendicular to both $(\gamma(s_{n\nu}),1)$ and $(\gamma'(s_{n\nu}),0)$ then $\langle v,\nabla\rangle U_\mu=O(2^{2n}r_1^2)$. Moreover, from (4.11)

$$\nabla_\xi s_{\rm cr}(\xi)=\frac{-\gamma'(s_{\rm cr}(\xi))}{\langle\gamma''(s_{\rm cr}(\xi)),\xi\rangle}$$

and thus $\langle v,\nabla\rangle s_{\rm cr}(\xi)=O(2^{n-k}r_1)$ if v is perpendicular to $(\gamma'(s_{n\nu}),0)$. Given the bounds on the directional derivatives of $s_{\rm cr}$ and U_μ the verification of (4.14) is straightforward.

Next to see (iv) observe that $|\omega_2^\mu|\le 2^{k+3}r_0^2$ on the support of a^μ (and hence on the support of $a^\mu_{n\nu}$). But we also have $|U_\mu|\ge 2^{k+2n-3}r_1^2$ and

$$(\omega_2^\mu)^2/|\omega_3^\mu|\le 2^{k+3}|s-s_{\rm cr}(\xi)|^2\le 2^{k+3}(2^{-k-8}|U_\mu|)$$

and thus $|\omega_2^\mu|\ge|U_\mu|/2\ge 2^{k+2n-4}r_1^2$. This forces $2^nr_1\le 2^4r_0$ if the support of $a^\mu_{n\nu}$ is nonempty.

Next consider the support of $b^\mu_{n,\nu}$ where $|s_{\rm cr}(\xi)-s^\mu|\le 2r_0$ and also $(s_{\rm cr}(\xi)-s^\mu)^2\ge 2^{-k-9}|U_\mu(\xi,\tau)|$; moreover $\max\{(s-s_{\rm cr}(\xi))^2,2^{-k}|U_\mu(\xi,\tau)\}\ge 2^{2n-3}r_1^2$. These conditions imply that $2^{2n-3}r_1^2\le 2^9(2r_0)^2$ if the support of $b^\mu_{n,\nu}$ is nonempty and thus (v) follows.

To see (vi) we set

$$\overline{\beta} = \overline{\beta}(\xi) = s_{\rm cr}(\xi) - s_{n\nu}$$

and observe that $|\overline{\beta}| \le 2^{n+1} r_1$ in the supports of $a^\mu_{n,\nu}$ and $b^\mu_{n,\nu}$, moreover $|\overline{\alpha}| \le 2r_0$. By (4.22) for $s = s^\mu$, we can write

$$\overline{\alpha} = \omega_1^\mu/\omega_3^\mu + \overline{\beta} + E \quad \text{ with } E = \mathcal{O}_1(24Mr_0^2). \tag{4.32}$$

Also $\omega_2^\mu(\xi,\tau) = U_\mu(\xi,\tau) + (\omega_1^\mu)^2/(2\omega_3^\mu)$ and

$$\begin{aligned}\langle u_2(\overline{\alpha}), \omega^\mu - \omega_3^\mu u_1(\overline{\alpha})\rangle &= \omega_1^\mu - \omega_3^\mu\overline{\alpha} + \omega_2^\mu\overline{\alpha} - \omega_3^\mu\overline{\alpha}^3/2 \\ &= \overline{\beta}(\omega_2^\mu - \omega_3^\mu) + \big[E(\omega_3^\mu - \omega_2^\mu) + \frac{\omega_2^\mu\omega_1^\mu}{\omega_3^\mu} - \omega_3^\mu\overline{\alpha}^3/2\big]\end{aligned}$$

and the expression [...] is easily seen to be $\mathcal{O}_1(100Mr_0^2)$ in view of the assumptions on the support of a^μ. Since we also assume $r_1 > 100Mr_0^{3/2}$ we deduce

$$|\langle u_2(\overline{\alpha}), \omega^\mu - \omega_3^\mu u_1(\overline{\alpha})\rangle| \le 2^{k+n+4} r_1.$$

Next we compute using (4.32)

$$\begin{aligned}\langle u_3(\overline{\alpha}), \omega^\mu\rangle &= -\omega_1^\mu\overline{\alpha} + \omega_3^\mu\frac{\overline{\alpha}^2}{2} + \frac{(\omega_1^\mu)^2}{2\omega_3^\mu} + U_\mu \\ &= U_\mu + \overline{\beta}^2\omega_3^\mu/2 + \omega_1^\mu E + \omega_3^\mu\overline{\beta}E + \omega_3^\mu E^2/2 \\ &= \mathcal{O}_1(2^{k+1}2^{2n}r_1^2) + \mathcal{O}_1(100M2^k r_0^3).\end{aligned}$$

which concludes the proof of (vi).

Now suppose (ξ,τ) belongs to the support of $a^\mu_{n,\nu}$. We have seen that then $|s_{\rm cr}(\xi) - s^\mu| \le 2r_0$. Since we assume that $\{s^\mu\}$ is a r_0-separated set we see that $(\xi,\tau) \in \operatorname{supp}(a^\mu_{n,\nu})$ for some n,ν can only happen for at most five μ. Exacly the same argument works for $b^\mu_{n\nu}$ in place of $a^\mu_{n\nu}$. Next, note that for a $\sigma \in \mathbb{R}$ one has $\eta_1(2^{2-2n}\sigma) \ne 0$ for at most five values of n. The definition of the functions $a^\mu_{n,\nu}$ and $b^\mu_{n,\nu}$ shows that for fixed μ there are at most five values of n for which $a^\mu_{n,\nu} \ne 0$ or $b^\mu_{n,\nu} \ne 0$. Finally, given u there are at most three values of ν for which $\zeta(u-\nu) \ne 0$. This shows that for fixed μ and fixed n there are at most three values of ν for which $a^\mu_{n,\nu} \ne 0$ or $b^\mu_{n,\nu} \ne 0$. A combination of these observations yields the assertion (vii). □

Lemma 4.2.6. *Suppose $Pl^n_{\mu\nu}$ is as in (4.27), and μ, n are fixed. Suppose that the Fourier transform of f_ν is supported in $Pl^n_{\mu\nu}$. Let $\mathfrak{I}^\mu_n = \{\nu : |s_{n\nu} - s^\mu| \le 2r_0\}$. Then for $\varepsilon > 0$, $p > p_W$,*

$$\Big\| \sum_{\nu\in\mathfrak{I}^\mu_n} f_\nu \Big\|_p \le \mathcal{C}_{p,\varepsilon}\Big(\frac{r_0}{2^n r_1}\Big)^{1-\frac{4}{p}+\varepsilon}\Big(\sum_\nu \|f_\nu\|_p^p\Big)^{1/p}.$$

Proof. Note that ω_μ in (4.26) is of rank three. Let $\varpi_\mu : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be an invertible map with $\varpi_i^\mu = \omega_i^\mu$ for $i = 1,2,3$. Let $g_\nu = |\det(\varpi^\mu)| f_\nu(\varpi^\mu \cdot)$. Then the Fourier transform of g_ν is supported in $R_\nu \times \mathbb{R}^{d+1-3}$ where the R_ν are C-extensions of plates in $\mathbb{R}^3$ associated to the curve $(\alpha, \alpha^2/2)$. Thus we can apply Wolff's theorem in three dimensions, in the form of Proposition 2.1, and obtain the estimate

$$\Big\| \sum_\nu g_\nu \Big\|_p \le \mathcal{C}_{p,\varepsilon}\Big(\frac{r_0}{2^n r_1}\Big)^{1-\frac{4}{p}+\varepsilon}\Big(\sum_\nu \|g_\nu\|_p^p\Big)^{1/p}$$

where the constant does not depend on μ. The assertion follows by rescaling. □

Lemma 4.2.7. *Suppose* $p \ge 2$, $r_1 \ge 2^{-k/2}$. *Then*

$$\Big(\sum_\nu \big\|\mathcal{T}[a^\mu_{0,\nu}, f]\big\|_p^p\Big)^{1/p} \le C r_1 \|f\|_p, \tag{4.33}$$

(with the usual sup *modification for* $p = \infty$*).*

Proof. We prove (4.33) by interpolation and it suffices to check the cases $p = 2$ and $p = \infty$. For $p = \infty$ the assertion follows from Lemma 4.2.1 and Lemma 4.2.2 if we observe that an additional s integration is extended over an interval of length $\mathcal{O}_1(r_1)$. For $p = 2$ we use van der Corput's lemma in the s variable with two derivatives to take advantage of our nondegeneracy hypothesis. Fix τ, ξ and observe that the amplitude of the oscillatory integral (as a function of s) is bounded and has an integrable derivative, with uniform bounds. Thus

$$\big|\mathfrak{m}[a^\mu_{0,\nu}](\xi,\tau)\big| \le C 2^{-k/2}$$

Observe that $|U_\mu| \le 2^k r_1^2$ on the support of $a^\mu_{0,\nu}$; moreover, the supports of the $a^\mu_{0,\nu}$ are essentially disjoint, by (vii) of Lemma 4.2.5. We obtain by Plancherel's theorem that

$$\begin{aligned}\sum_\nu \big\|\mathcal{T}[a^\mu_{0,\nu}, f]\big\|_2^2 &= c\sum_\nu \int_\xi \int_{\tau:|U_\mu(\xi,\tau)|\le 2^k r_1^2} \big|\mathfrak{m}[a^\mu_{0,\nu}](\xi,\tau)\big|^2 d\tau |\widehat{f}(\xi)|^2 d\xi \\ &\lesssim 2^{-k} 2^k r_1^2 \|f\|_2^2\end{aligned}$$

which is the desired bound for $p = 2$. □

Lemma 4.2.8. *For* $n \ge 1$, $2 \le p \le \infty$,

$$\Big(\sum_{\mu,\nu} \big\|\mathcal{T}[a^\mu_{n\nu}, f]\big\|_p^p\Big)^{1/p} \le C 2^{-k-n} r_1^{-1} \|f\|_p. \tag{4.34}$$

Proof. We argue similarly as in the proof of Lemma 4.2.7 but begin by integrating by parts with respect to t to get

$$\mathfrak{m}[a^\mu_{n,\nu}](\xi,\tau) = \iint e^{-it(\tau+\langle\gamma(s),\xi\rangle)} \frac{\partial_t a^\mu_{n,\nu}(s,t,\xi,\tau)}{i(\tau+\langle\gamma(s),\xi\rangle)} ds dt.$$

Now expand $\langle\gamma(s),\xi\rangle$ about $s_{\rm cr}(\xi)$ and by (4.23)

$$\tau + \langle\gamma(s),\xi\rangle = U_\mu(\xi,\tau) + \mathcal{O}_1(2^k(s-s_{\rm cr}(\xi)^2) + \mathcal{O}_1(2^{k+10} M r_0^3)$$

in the support of $a^\mu_{n,\nu}$. Since $n \ge 1$ one also has $|s - s_{\rm cr}(\xi)|^2 \le 2^{-k-8}|U_\mu|$ and hence

$$|\tau + \langle\gamma(s),\xi\rangle| \ge \frac{1}{2}|U_\mu(\xi,\tau)| \ge 2^{k+2n-4} r_1^2.$$

Consequently, the multiplier $(\xi,\tau) \mapsto \partial_t a^\mu_{n,\nu}(s,t,\xi,\tau)(\tau+\langle\gamma(s),\xi\rangle)^{-1}$ can be written as $C2^{-(k+2n)} r_1^{-2}$ times a multiplier in $\mathcal{S}_k(r,s)$ and thus Lemma 4.2.2 applies. Since we perform an s-integration over an interval of length $O(2^n r_1)$ we get the asserted $\ell^\infty(L^\infty)$ bound.

For the $\ell^2(L^2)$ estimate we apply van der Corput's Lemma with second derivatives and check using the support properties of $a^\mu_{n,\nu}$ that the L^∞ norm of the amplitude and the L^1 norm (in s) of its derivative is bounded by $C2^{-k+2n}r_1^{-2}$. Thus we now obtain

$$\sum_{\mu,\nu}\big\|\mathcal{T}[a^\mu_{n,\nu},f]\big\|_2^2 \lesssim \sum_{\mu,\nu}\int_\xi\int_{|U_\mu(\xi,\tau)|\le 2^{k+2n}r_1^2} |2^{-k/2}(2^{k+2n}r_1^2)^{-1}|^2 d\tau|\widehat{f}(\xi)|^2 d\xi$$
$$\lesssim \big(2^{-k-n}r_1^{-1}\|f\|_2\big)^2.$$

□

Lemma 4.2.9. *For $n\ge 1$, $2\le p\le\infty$,*

$$\Big(\sum_{\mu,\nu}\big\|\mathcal{T}[b^\mu_{n,\nu},f]\big\|_p^p\Big)^{1/p} \le C2^{-k-n}r_1^{-1}\|f\|_p. \tag{4.35}$$

Proof. We argue similarly as in Lemma 4.2.8. Now we integrate by parts in s to see that

$$\mathfrak{m}[b^\mu_{n,\nu}](\xi,\tau) = \iint e^{-it(\tau+\langle\gamma(s),\xi\rangle)} c^\mu_{n,\nu}(s,t,\xi,\tau)dsdt$$

where

$$c^\mu_{n,\nu}(s,t,\xi,\tau) = \frac{\partial_s b^\mu_{n,\nu}(s,t,\xi,\tau)}{it\langle\gamma'(s),\xi\rangle} + \frac{b^\mu_{n,\nu}(s,t,\xi,\tau)\langle\gamma''(s),\xi\rangle}{it(\langle\gamma'(s),\xi\rangle)^2}.$$

Now

$$|\langle\gamma'(s),\xi\rangle| \approx 2^k|s-s_{\rm cr}(\xi)| \approx 2^k2^nr_1$$

on the support of $b^\mu_{n,\nu}$.

The multiplier $(\xi,\tau)\mapsto c^\mu_{n,\nu}(s,t,\xi,\tau)$ is $C2^{-(k+2n)}r_1^{-2}$ times a multiplier in $\mathcal{S}_k(r,s)$. Thus Lemma 4.2.2 applies and the $\ell^\infty(L^\infty)$ estimate follows in the same way as in Lemma 4.2.8.

For the L^2 estimate we may integrate by parts in t and obtain

$$|\mathfrak{m}[b^\mu_{n,\nu}](\xi,\tau)| \lesssim \iint \min\{1,|\tau+\langle\gamma(s),\xi\rangle|^{-1}\}|c^\mu_{n,\nu}(s,t,\xi,\tau)|dtds,$$

and consequently $\sum_{\mu,\nu}\|\mathcal{T}[b^\mu_{n,\nu},f]\|_2^2$ is dominated by

$$\sum_{\mu,\nu}\iint_{supp(b^\mu_{n,\nu})}\Big[\int_{\substack{|s-s_{\rm cr}(\xi)|\\ \le 2^nr_1}} \frac{2^{-k-2n}r_1^{-2}}{1+|\tau+\langle\gamma(s),\xi\rangle|}ds\Big]^2 d\tau|\widehat{f}(\xi)|^2d\xi$$
$$\le 2^{n+1}r_1\sup_\xi\Big[\int_{\substack{|s-s_{\rm cr}(\xi)|\\ \le 2^nr_1}}\int_\tau \frac{2^{-2k-4n}r_1^{-4}}{(1+|\tau+\langle\gamma(s),\xi\rangle|)^2}d\tau ds\Big]^2 \sum_{\mu,\nu}\iint_{supp(b^\mu_{n,\nu})}\int|\widehat{f}(\eta)|^2d\eta$$

which is bounded by $(2^{-k-n}r_1^{-1}\|f\|_2)^2$. An interpolation yields the claimed inequality for $2\le p\le\infty$. □

Conclusion of the proof of Proposition 4.2.3. It is immediate from the decomposition (4.25), and Lemma 4.2.6 that for all $\varepsilon\in(0,1)$, $p>p_W$,

$$\Big(\sum_\mu\big\|\mathcal{T}[a^\mu,f]\big\|_p^p\Big)^{1/p} \le \mathcal{C}_{p,\varepsilon}(r_0/r_1)^{1-\frac4p+\varepsilon}\Big[\Big(\sum_\nu\big\|\mathcal{T}[a^\mu_{0,\nu},f]\big\|_p^p\Big)^{1/p}$$
$$+\Big(\sum_{\substack{n\ge1\\ 2^nr_1\le r_0}}\sum_\nu\big\|\mathcal{T}[a^\mu_{n,\nu},f]\big\|_p^p+\big\|\mathcal{T}[b^\mu_{n,\nu},f]\big\|_p^p\Big)^{1/p}\Big]$$

and we can apply Lemma 4.2.8 and Lemma 4.2.9 to the terms involving $n \ge 1$. Thus the left hand side of the inequality is dominated by

$$\mathcal{C}_{p,\varepsilon}(r_0/r_1)^{1-\frac{4}{p}+\varepsilon}\Big[\Big(\sum_\nu \big\|\mathcal{T}[a^\mu_{0,\nu},f]\big\|_p^p\Big)^{1/p} + 2^{-k}r_1^{-1}\|f\|_p\Big].$$

There is a constant C so that the symbols $C^{-1}a^\mu_{0,\nu}$ belong to $\mathfrak{S}_k(r_1, s_{0\nu})$. Moreover, given fixed ν the function $a^\mu_{0,\nu}$ is not identically 0 for at most five μ and the $s_{0\nu}$ are r_1-separated. By a pidgeonhole argument we deduce the assertion of the proposition. □

Conclusion of the proof of Theorem 4.2. Let $\varepsilon_1 = \varepsilon_0^2/(dM)$. For fixed s_0 and we shall prove an estimate for the symbol $\widetilde{b}(s,t,\xi) = b(s,t,\xi)\chi(2^{k\varepsilon_1}(s-s_0))$, namely

$$\|\mathfrak{A}_\gamma[\widetilde{b},f]\|_{L^p(\mathbb{R}^{d+1})} \le C_p(\varepsilon_0)2^{-k(\frac{2}{p}-\frac{\varepsilon_0}{2})}\|f\|_p, \quad p > p_W. \tag{4.36}$$

The assertion of te theorem follows from (4.36) as b is a sum of $O(2^{k\varepsilon_1})$ such symbols.

We write by using the Fourier inversion formula in $\mathbb{R}^{d+1}$

$$\mathfrak{A}_\gamma[\widetilde{b},f] = \sum_{\ell=0}^\infty \mathcal{T}[\widetilde{a}_\ell, f]$$

where

$$\widetilde{a}_0(s,t,\xi,\tau) = \widetilde{b}(s,t,\xi)\eta_0(2^{2k\varepsilon_1}(2^{-k}|\tau + \langle\gamma(s_{\rm cr}(\xi)),\xi\rangle| + (s-s_{\rm cr}(\xi))^2))$$

and, for $\ell \ge 1$,

$$\widetilde{a}_\ell(s,t,\xi,\tau) = \widetilde{b}(s,t,\xi)\eta_1(2^{2k\varepsilon_1-2\ell+2}(2^{-k}|\tau + \langle\gamma(s_{\rm cr}(\xi)),\xi\rangle| + (s-s_{\rm cr}(\xi))^2)).$$

We first show the main estimate which is

$$\|\mathcal{T}[\widetilde{a}_0,f]\|_{L^p(\mathbb{R}^{d+1})} \le C_p(\varepsilon_0)2^{-k(2/p-\varepsilon_0/2)}\|f\|_{L^p(\mathbb{R}^d)}, \quad p > p_W. \tag{4.37}$$

Now for a constant C we have $C^{-1}\widetilde{a}_0 \in \mathfrak{S}_k(2^{-k\varepsilon_1}, s_0)$. We apply Proposition 4.2.3 iteratively choosing r_0, r_1 to be

$$r_0(n) = (2^{-k\varepsilon_1}M)^{(3/2)^n}, \quad r_1(n) = (2^{-k\varepsilon_1}100M)^{(3/2)^{n+1}},$$

for $n = 0,\dots,N$, where $N = N(\varepsilon_1)$ is the largest integer for which $r_1(n) \ge 2^{-k(\frac{1}{2}-\varepsilon_1)}$. Thus certainly $r_1(N) \le 2^{-\frac{k}{2}+2k\varepsilon_1}$ and $N = N(\varepsilon_1) \le C/\varepsilon_1 \le C'/\varepsilon_0^2$.

By Proposition 4.2.3 we obtain for all $p > p_W$, $\varepsilon > 0$ that

$$\begin{aligned}\big\|\mathcal{T}[\widetilde{a}_0,f]\big\|_p \le (\mathcal{C}_{p,\varepsilon})^N r_1(N)^{-(1-4/p+N\varepsilon)}\Big(\sum_\nu \|\mathcal{T}[a_\nu,f]\|_p^p\Big)^{1/p} \\ + 2^{-k}\sum_{n=0}^N (\mathcal{C}_{p,\varepsilon})^n r_1(n)^{4/p-2-n\varepsilon}\|f\|_p\end{aligned} \tag{4.38}$$

where $a_\nu \in \mathfrak{S}_k(s_\nu, r_1(N))$ and the s_ν are $r_1(N)$-separated points. Note that since $r_1(N) \ge 2^{-k/2}$

$$2^{-k}\sum_{n=0}^N r_1(n)^{4/p-2-n\varepsilon} \le C2^{-k(\frac{2}{p}-N\varepsilon)}(2^k r_1^2)^{-(1-\frac{2}{p})}; \tag{4.39}$$

moreover by Lemma 4.2.7

$$\Big(\sum_\nu \big\|\mathcal{T}[a_\nu,f]\big\|_p^p\Big)^{1/p} \lesssim r_1(N)\|f\|_p. \tag{4.40}$$

We choose $\varepsilon = (\varepsilon_1/(1000C))^2$ in (4.38) which makes $N\varepsilon \ll \varepsilon_0/4$ and still, by our previous choice of ε_1, the resulting constant $(\mathcal{C}_{p,\varepsilon})^N$ depends only on ε_0 and p. We combine the resulting bound with (4.39) and (4.40) and the main estimate (4.37) follows.

To finish the proof we have to dispose of the terms $\mathcal{T}[a_\ell,\cdot]$ for $\ell \ge 1$; these are error terms which can be handled by standard arguments. We split (in analogy to a previous decomposition) $\widetilde{a}_\ell = \widetilde{a}_{\ell,1} + \widetilde{a}_{\ell,2}$ where

$$\widetilde{a}_{\ell,1}(s,t,\xi,\tau) = \widetilde{a}_\ell(s,t,\xi,\tau)\eta_0\big(M\frac{(s-s_{\rm cr}(\xi))^2}{2^{-k}|\tau+\langle\gamma(s_{\rm cr}(\xi)),\xi\rangle|}\big),$$
$$\widetilde{a}_{\ell,2}(s,t,\xi,\tau) = \widetilde{a}_\ell(s,t,\xi,\tau)\big(1-\eta_0\big(M\frac{(s-s_{\rm cr}(\xi))^2}{2^{-k}|\tau+\langle\gamma(s_{\rm cr}(\xi)),\xi\rangle|}\big)\big);$$

note that $\widetilde{a}_{\ell,2} = 0$ if $\ell \gg 2^{k\varepsilon_1}$. We use an integration by parts in t for the integral defining $\mathfrak{m}[\widetilde{a}_{\ell,1}]$ and an integration by parts in s for the integral defining $\mathfrak{m}[\widetilde{a}_{\ell,2}]$.

Now for $i=1,2$

$$\mathcal{T}[\widetilde{a}_{\ell,i},f](x,t') = \int\int K_s[m_{s,\ell,i}](x-y,t')ds\, f(y)dy$$

where we use the notation (4.15) with

$$m_{s,\ell,1}(t,\xi,\tau) = \frac{\widetilde{a}_{\ell,1}(s,t,\xi,\tau)}{i(\tau+\langle\gamma(s),\xi\rangle)},$$
$$m_{s,\ell,2}(t,\xi,\tau) = \Big[\frac{\partial_s\widetilde{a}_{\ell,2}(s,t,\xi,\tau)}{it\langle\gamma'(s),\xi\rangle} + \frac{\widetilde{a}_{\ell,2}(s,t,\xi,\tau)\langle\gamma''(s),\xi\rangle}{it(\langle\gamma'(s),\xi\rangle)^2}\Big].$$

We argue as in the proof of Lemma 4.2.2 and by a straightforward integration by parts we obtain the bounds

$$|K_s[m_{s,\ell,i}](x,t')|$$
$$C2^{-k(1-2\varepsilon_1)-\ell}\int \frac{2^{kd}2^\ell}{(1+2^{k(1-2\varepsilon_1)+\ell}|t-t'|+2^{k(1-2\varepsilon_1)}|x-t'\gamma(s)|)^{d+2}}\,dt$$

for $i=1,2$; here we use for the second kernel that $m_{s,\ell,2}=0$ for $\ell \ge 2^{2k\varepsilon_1}$.

This estimate implies (after an integration in s) that the terms involving $\widetilde{a}_\ell$ for $\ell > 0$ are error terms and we get the estimates

$$\|\mathcal{T}[\widetilde{a}_\ell,f]\|_{L^p(\mathbb{R}^{d+1})} \lesssim 2^{-k(1-2\varepsilon_1(d+1))}\|f\|_{L^p(\mathbb{R}^d)}$$

for $1 \le p \le \infty$ and of course the constant here is much smaller than $2^{-2k/p}$ for $p > 4$ and in particular for $p > p_W$. This finishes the proof of Theorem 4.2. □

4.3. **Proof of Theorem 4.1.** We may assume that $B > 100(d+\delta^{-1})$. In addition by a reparametrization we may also assume that Γ is parametrized by arclength s (consequently we may have to replace B by a power of B).

We localize in s (splitting the parameter interval in $O(B^{102})$ pieces) and assume that the symbol is localized to an s-interval $I(s_0)$ centered at s_0, and of length $\le B^{-100}$. By further localization in Ω we split the symbol into $O(B^{100d})$ pieces localized in balls of the form $\Omega(\xi^0) = \{\xi : |\xi-\xi^0| \le 2^kB^{-10}\}$ where $B^{-1}2^k \le |\xi^0| \le$

$2^k B$. We now assume that our symbol a is supported in $I(s_0)\times[1,2]\times\Omega(\xi^0)$ and that a satisfies differentiability conditions similar to (4.3), but with the constant $\mathcal{C}[a]$ replaced by $C_d\mathcal{C}[a]B^{1000d}$; moreover we assume the lower bound

$$\big|\langle\Gamma'(s),\xi\rangle\big|+\big|\langle\Gamma''(s),\xi\rangle\big|\ge 2B^{-2}|\xi| \quad \text{if } (s,t,\xi)\in\operatorname{supp}(a). \tag{4.41}$$

We set $\theta=\xi^0/|\xi^0|$ and distinguish two cases. In the first case we assume that $|\langle\Gamma'(s_0),\theta^0\rangle|\ge B^{-100}$; then by the support properties after localization $|\langle\Gamma'(s),\xi\rangle|\ge B^{-90}|\xi|$ on the support of a. This allows us to perform an integration by parts in s first, thus gaining a power of 2^k and standard estimates yield that in the present case the $L^p(\mathbb{R}^d)$ norm of $\mathfrak{A}_\Gamma[a,f](\cdot,t)$ is bounded by $C_{B,d}2^{-k}\|f\|_p$ for $1\le p\le\infty$, uniformly in $t\in[1,2]$. Thus in this case we obtain a better bound than the one claimed in (4.5).

For the second (main) case we have the inequalities

$$|\langle\Gamma'(s_0),\theta^0\rangle|\le B^{-100}, \tag{4.42}$$

$$|\langle\Gamma''(s_0),\theta^0\rangle|\ge B^{-2}. \tag{4.43}$$

Now let $\{v_1,\dots,v_d\}$ be an orthonormal basis of $\mathbb{R}^d$ so that $v_1=\Gamma'(s_0)$ and span $\{v_1,v_2\}=\operatorname{span}\{\Gamma'(s_0),\theta^0\}$. Let L be the linear transformation with $L(v_i)=v_i$ for $i=1$, $3\le i\le d$ and $L(v_2)=\langle\Gamma''(s_0),v_2\rangle^{-1}v_2$. Let $\widetilde\Gamma(s)=L\Gamma(s)$.

By (4.42) and (4.43) and a Taylor expansion $|\widetilde\Gamma'(s)|=1+O(B^{-10})$ and since we assume that Γ is parametrized by arclength we have $\langle\Gamma'(s),\Gamma''(s)\rangle=0$. A calculation shows that $\langle\widetilde\Gamma''(s),\theta^0\rangle=1+O(B^{-10})$.

Notice that

$$\mathfrak{A}_\Gamma[a,f](x,t)=\mathfrak{A}_{\widetilde\Gamma}[\widetilde a,f\circ L^{-1}](Lx,t) \tag{4.44}$$

where $\widetilde a(s,t,\eta)=a(s,t,L^t\eta)$. After a reparametrization of $\widetilde\Gamma$ by arclength an application of Theorem 4.2 shows that

$$\big\|\mathfrak{A}_{\widetilde\Gamma}[\widetilde a,f]\big\|_{L^p(\mathbb{R}^{d+1})}\le C(\varepsilon_0,p,B)\mathcal{C}[a]2^{-k(\frac{2}{p}-\varepsilon_0)}\|f\|_{L^p(\mathbb{R}^d)},\quad p>p_W,$$

and the corresponding assertion for $\mathfrak{A}_\Gamma$ follows by (4.44). □

5. Local Smoothing for Curves in $\mathbb{R}^3$

We now return to the situation in $\mathbb{R}^3$ and consider *curves with nonvanishing curvature and torsion.* We shall use notation as in §3 and prove an estimate for the t-dilates of the operators $\mathcal{A}^{k,l,\nu}$ defined in (3.16). The following lemma is proved by rescaling and the results of the previous sections. Define

$$\widehat{\mathcal{A}_t^{k,l,\nu}f}(\xi)=m_k[a_{k,l,\nu}(t\cdot)]\widehat f(\xi) \tag{5.1}$$

and let χ be a smooth function supported in $(1/2,2)$.

Proposition 5.1. *For $p>p_W$, $l<k/3$, $\varepsilon>0$,*

$$\Big(\int\|\chi(t)\mathcal{A}_t^{k,l,\nu}f\|_p^p dt\Big)^{1/p}\le C_\varepsilon 2^{-l(1-6/p)}2^{-2k/p}2^{k\varepsilon}\|f\|_p.$$

Proof. The symbol $a_{k,l,\nu}$ in (5.1) is supported in a set where $|\langle\xi,T(s_\nu)\rangle|\approx 2^{k-2l}$, $|\langle\xi,N(s_\nu)\rangle|\lesssim 2^{k-l}$, $|\langle\xi,B(s_\nu)\rangle|\approx 2^k$. We shall rescale the parameter $s=s_\nu+2^{-l}u$ with $u\lesssim 1$, moreover we rescale in ξ as follows. Let U_ν be the rotation which maps the unit vectors e_1, e_2, e_3 to $T(s_\nu), N(s_\nu), B(s_\nu)$. Let $\Delta_l\eta = (2^l\eta_1, 2^{2l}\eta_2, 2^{3l}\eta_3)$ and let $L_{l,\nu} = U_\nu \circ \Delta_l$. Then

$$(u,\eta) \to c_{k,l,\nu}(u,\eta) := a_{k,l,\nu}(s_\nu + 2^{-l}u, L_{l,\nu}\eta)$$

is supported in a set where $|\eta| \approx 2^{k-3l}$ and $|u| \lesssim 1$ and there are the estimates

$$\big|\partial_u^{(n)}\partial_\eta^\alpha c_{k,l,\nu}(u,\eta)\big| \le C_{n,\alpha}2^{-(k-3l)|\alpha|}.$$

Moreover if we set

$$\Gamma_{l,\nu}(u) = L^*_{l,\nu}\gamma(s_\nu + 2^{-l}u)$$

then $\Gamma_{l,\nu}$ is a C^5 curve with upper bounds uniformly in l, ν and we also have

$$|\langle \Gamma''_{l,\nu}(u), \eta\rangle| \approx |\eta| \approx 2^{k-3l}$$

in the support of $c_{k,l,\nu}$ (again with the implicit constants uniform in ℓ, ν).

Changing variables we get

$$\begin{aligned}(2\pi)^d \mathcal{A}_t^{k,l,\nu} f(x) &= 2^{-l}\iint e^{it\langle\gamma(s_\nu+2^{-l}u),\xi\rangle + i\langle x,\xi\rangle} a(s_\nu + 2^{-l}u, \xi)\widehat{f}(\xi)d\xi du\\ &= 2^{-l}\int e^{it\langle\Gamma_{l,\nu}(u),\eta\rangle + i\langle L^*_{l,\nu}x,\eta\rangle} c_{k,l,\nu}(\eta)\widehat{f}(L_{l,\nu}\eta)\, 2^{6l} d\eta\, du\\ &= 2^{-l}T_t^{k,l,\nu}[f(L^{*}_{l,\nu}{}^{-1}\cdot)](L^*_{l,\nu}x) \qquad (5.2)\end{aligned}$$

where

$$T_t^{k,l,\nu}g(x) = \iint e^{it\langle\Gamma_{l,\nu}(u),\eta\rangle} c_{k,l,\nu}(u,\eta)\widehat{g}(\eta)e^{i\langle x,\eta\rangle}d\eta du.$$

Thus we can apply Theorem 4.1 for the dyadic annulus of width 2^{k-3l}, and obtain

$$\Big(\int \big\|\chi(t)T_t^{k,l,\nu}g\big\|_p^p dt\Big)^{1/p} \le C_\varepsilon 2^{-2(k-3l)/p}2^{(k-3l)\varepsilon}\|g\|_p.$$

We rescale using (5.2) to obtain the asserted bound. □

Proof of Theorem 1.4. We apply inequality (3.23), rescaled by the t-dilation, and combine it with Proposition 5.1 to obtain

$$\begin{aligned}\Big(\int \big\|\chi(t)\sum_\nu \mathcal{A}_t^{k,l,\nu}f\big\|_p^p dt\Big)^{1/p} &\le C_\varepsilon 2^{2l(\frac12-\frac2p+\varepsilon)}\Big(\sum_\nu\int \big\|\chi(t)\mathcal{A}_t^{k,l,\nu}f\big\|_p^p dt\Big)^{1/p}\\ &\le C'_\varepsilon 2^{-k(\frac2p-\varepsilon)}2^{-2l(\frac1p-\varepsilon)}\|f\|_p\end{aligned}$$

and thus

$$\Big(\int \big\|\chi(t)\mathcal{A}_t^{k,l}f\big\|_p^p dt\Big)^{1/p} \le C_\varepsilon 2^{-2(k-l)/p+2k\varepsilon}\|f\|_p, \quad p > p_W. \qquad (5.3)$$

This is the main estimate and we may sum over $l < k/3$. There are similar estimates for the operators $\sum_\nu \widetilde{\mathcal{A}}_t^{k,\nu}$ and $\sum_\nu \mathcal{B}_t^{k,l,\nu}$ obtained if we scale by t in the definitions (3.16), (3.17); however these follow already by integrating out the fixed time estimates implied by Proposition 3.1. The conclusion is that if m_k is as in (3.4) then

$$\Big(\int \|\chi(t)\mathcal{F}^{-1}[m_k(t\cdot)\widehat{f}]\|_p^p dt\Big)^{1/p} \le C_\varepsilon 2^{-k(\frac{4p}{3}-\varepsilon)}\|f\|_p$$

and the assertion of Theorem 1.4 on boundedness in Sobolev spaces follows by standard arguments. □

6. Maximal Functions

Proof of Theorem 1.2. Given Theorem 1.4 the proof is straightforward for the case of curves with nonvanishing curvature and torsion. Let $\mathcal{L}_k$ a Littlewood-Paley operator which localizes to frequencies of size $\approx 2^k$. Then for $p > p_W$

$$\Big(\int_1^2 \big\|\mathcal{A}_{2^\ell t}\mathcal{L}_{k+\ell} f\big\|_p^p dt\Big)^{1/p} \le C_{\varepsilon,p} 2^{-k(\frac{4}{3p}-\varepsilon)}\|\mathcal{L}_{k+\ell}f\|_p,$$

and

$$\Big(\int_1^2 \big\|(\partial/\partial t)\mathcal{A}_{2^\ell t}\mathcal{L}_{k+\ell} f\big\|_p^p dt\Big)^{1/p} \le C_{p,\varepsilon} 2^{-k(\frac{4}{3p}-\varepsilon)}(1+2^k\sup_{s\in I}|\gamma(s)|)\|\mathcal{L}_{k+\ell}f\|_p,$$

and by standard arguments we obtain

$$\begin{aligned}&\big\|\sup_{\ell\in\mathbb{Z}}\sup_{1\le t\le 2}|\mathcal{A}_{2^\ell t}\mathcal{L}_{k+\ell}f|\big\|_p \\ &\qquad\le C_{p,\varepsilon}2^{-k(\frac{4}{3p}-\varepsilon)}\big(1+2^k\sup_{s\in I}|\gamma(s)|\big)^{1/p}\Big(\sum_{\ell\in\mathbb{Z}}\|\mathcal{L}_{k+\ell}f\|_p^p\Big)^{1/p}, \quad p>p_W.\end{aligned}$$

Since $p\ge 2$ we have $(\sum_{\ell\in\mathbb{Z}}\|\mathcal{L}_{k+\ell}f\|_p^p)^{1/p}\lesssim\|f\|_p$. Similar L^2 estimates based on van der Corput's lemma yield

$$\big\|\sup_{\ell\in\mathbb{Z}}\sup_{1\le t\le 2}|\mathcal{A}_{2^\ell t}\mathcal{L}_{k+\ell}f|\big\|_2 \le C2^{-k/3}\big(1+2^k\sup_{s\in I}|\gamma(s)|\big)^{1/2}\|f\|_2$$

and an interpolation shows that

$$\big\|\sup_{\ell\in\mathbb{Z}}\sup_{1\le t\le 2}|\mathcal{A}_{2^\ell t}\mathcal{L}_{k+\ell}f|\big\|_p \lesssim C_p(1+\sup_{s\in I}|\gamma(s)|)^{1/p}2^{-ka(p)}\|f\|_p \tag{6.1}$$

with $a(p)>0$ if $p>(p_W+2)/2$. This proves the statement of Theorem 1.2 in the case of curves with nonvanishing curvature and torsion.

In the finite type case we use rescaling as in §3. We may after using a partition of unity assume that (3.41) holds, with $n_1<n_2<n_3$, and $n_3\le n$. Then, with $A_{j,t}$ as in (3.42) we need to show that

$$\big\|\sup_{\ell}\sup_{1\le t\le 2}|A_{j,2^\ell t}f|\big\|_p \lesssim 2^{j(\frac{n_3}{p}-1)}\|f\|_p, \quad p>(p_W+2)/2. \tag{6.2}$$

We may apply (6.1) to the normalized curves $\delta_j\gamma(s_0)+\Gamma_j$ (where Γ_j is as in (3.43)), and observe that

$$\sup_u|\delta_j\gamma(s_0)+\Gamma_j(u)| = O(2^{n_3}).$$

Thus using also (3.44) and setting $f_{-j}=f\circ\delta_{-j}$, $N=n_1+n_2+n_3$, we see that $\|\sup_\ell\sup_{1\le t\le 2}|A_{j,2^\ell t}f|\|_p$ is controlled by

$$\begin{aligned}&2^{-j}\Big\|\sup_{\ell}\sup_{1\le t\le 2}\Big|\int f_{-j}(\delta_j\cdot -2^\ell t\delta_j\gamma(s_0)-2^\ell t\Gamma_j(u))\chi_j(u)du\Big|\Big\|_p\\ &\lesssim 2^{j(\frac{n_3}{p}-1)}2^{-N/p}\|f_{-j}\|_p \lesssim 2^{j(\frac{n_3}{p}-1)}\|f\|_p,\end{aligned}$$

and obtain (6.2). We need to sum in j in (6.2) which is possible since also $p>n_3$. □

A two-parameter maximal function. Our results on local smoothing can also be used to prove bounds for certain two-parameter maximal functions. Consider the two-parameter family of helices

$$H(a,b) := \{\gamma_{a,b} = (a\cos(2\pi s),\, a\sin(2\pi s),\, bs) \,:\, 0 \le s \le 1\}\,,\ 1 < a,b < 2.$$

Then we obtain a lower bound on the Hausdorff dimension of some "Kakeya-type" sets.

Proposition 6.1. *Let F be a set which for every $x \in \mathbb{R}^3$ contains a helix $x+H(a,b)$ for some (a,b), $1 < a,b < 2$. Then the Hausdorff dimension of F is at least 8/3.*

By arguments in [2] one sees that Proposition 6.1 is a consequence of an estimate for a local maximal operator, namely

$$\Big\| \sup_{1\le a,b\le 2} \Big| \int f(x-\gamma_{a,b}(s))\chi(s)ds\Big|\Big\|_p \le C_\alpha \|f\|_{L^p_\alpha}, \quad p > p_W, \alpha > (3p)^{-1}. \tag{6.3}$$

Proof. We only sketch the argument since it follows the same lines as the one in the proof of Theorem 1.2, however it uses as an additional ingredient the relation between $\partial_a\gamma_{a,b}$ and γ''.

Let $\widetilde{\mathcal{A}}^k_{a,b}$, $\mathcal{A}^{k,l}_{a,b}$, $\mathcal{B}^{k,l}_{a,b}$ be the operators with symbols $m_k(\widetilde{a}_k)$, $m_k(a_{k,l})$, $m_k(b_{k,l})$ as in (3.6), (3.7), for the curve $\gamma_{a,b}$. (6.3) follows from

$$\Big\| \sup_{1\le a,b\le 2} |\mathcal{A}^{k,l}_{a,b} f|\Big\|_p \le C_\varepsilon 2^{l/p} 2^{k\varepsilon}\|f\|_p, \quad p > p_W, \quad l < k/3, \tag{6.4}$$

and related statements for $\widetilde{\mathcal{A}}^k_{a,b}$, $\mathcal{B}^{k,l}_{a,b}$. By standard arguments the proof of (6.4) can be reduced to

$$\Big(\iint_{1\le a,b\le 2} \Big\| \frac{\partial^{j_1+j_2}}{(\partial a)^{j_1}(\partial b)^{j_2}} \mathcal{A}^{k,l}_{a,b} f\Big\|_p^p dadb\Big)^{1/p} \le C_\varepsilon 2^{(k-l)j_1+kj_2} 2^{-2(k-l)/p+k\varepsilon}\|f\|_p, \tag{6.5}$$

$l < k/3$. When $j_1 = 0$, $j_2 \in \{0,1\}$ inequality (6.5) follows from (5.3). For the a-differentiation ($j_1 = 1$) inequality (6.5) asserts a blowup of merely 2^{k-l}. This happens because the a-differentiation of the phase yields an additional factor of

$$\partial_a\langle\gamma_{a,b}(s),\xi\rangle = \xi_1\cos(2\pi s) + \xi_2\sin(2\pi s) = -(4\pi^2 a)^{-1}\langle\gamma''_{a,b}(s),\xi\rangle,$$

for the symbol, and $\langle\gamma''_{a,b}(s),\xi\rangle$ is of size $\approx 2^{k-l}$ on the support of $m_k(a_{k,l})$. It is here where we use the improvements stated in part (iv) of Lemma 3.2 and part (iii) of Lemma 3.3. □

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