L^p SOBOLEV REGULARITY OF A RESTRICTED X-RAY TRANSFORM IN R^3

MALABIKA PRAMANIK AND ANDREAS SEEGER

Abstract. We consider the $L^p \to L^p_\sigma$ boundedness of a model restricted X-ray transform in $\mathbb{R}^3$, associated to a rigid line complex. We discuss some necessary conditions and, assuming a finite type condition, we show that the sharp $L^p \to L^p_{1-1/p}$ result holds for $p > 1$ close to 1.

1. Introduction

Let I be a compact interval and suppose that $\gamma : I \to \mathbb{R}^2$ be a smooth regular curve (i.e. we assume $\gamma'(s) \neq 0$). We say for $m \geq 2$ that γ is of type m at s if γ has contact of order m with its tangent line at $\gamma(s)$. The maximal order of contact in I is referred to as the maximal type of γ in I. For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$ define

$$\mathcal{R}f(x', \alpha) = \chi_1(\alpha) \int_1^2 f(x' + s\gamma(\alpha), s)\, \chi_2(s)\, ds, \tag{1.1}$$

where $x' = (x_1, x_2)$, and χ_1 and χ_2 are smooth real valued functions supported in the interior of I and $[1, 2]$ respectively. We shall assume that χ_2 is nonnegative. The operator $\mathcal{R}$ exhibits a partial translation invariance; i.e. $\mathcal{R}f(x'+z', \alpha) = \mathcal{R}[f(\cdot+z', \cdot)](x', \alpha)$. It serves as a model case for a more general class of restricted X-ray transforms considered in [3], [6], [7], [8], [4] and elsewhere. In the well-curved case (*i.e.* $m = 2$) these operators are Fourier integral operators with one-sided fold singularities.

Supported in part by grants from the US National Science Foundation. The second named author would like to thank the organizers of the HAAO conference for their support and hospitality.

We are interested in the $L^p \to L^p_\sigma$ mapping properties of $\mathcal{R}$ (here L^p_σ denotes the standard L^p-Sobolev space). Necessary and sufficient conditions for L^2-Sobolev inequalities of general restricted X-ray transforms in the well-curved case are contained in [7], [8], [4]; $\mathcal{R}$ maps L^2 to $L^2_{1/4}$ and this is best possible. For results on L^2 boundedness in the finite type case see [11].

Concerning the L^p Sobolev regularity we we first state some necessary conditions.

Proposition 1.1. *Suppose that the cutoff functions χ_1 and χ_2 are not identically zero and suppose that there is α with $\chi_1(\alpha) \neq 0$ and γ is of type m at α. Suppose that $\mathcal{R}$ maps L^p boundedly to L^p_σ. Then*

$$\sigma \le \min\{1 - \frac{1}{p}, \frac{1}{4}, \frac{1}{mp}\}. \tag{1.2}$$

The proposition will be proved in §2. We remark that the condition $\sigma \le \min\{1 - 1/p, 1/mp\}$ is essentially known; see [7] for an example in the well curved case (where $\gamma'' \neq 0$) and [11] for the finite type case; a simpler argument is given in §2.2 below. The condition $\sigma \le 1/4$ seems to be not have been observed in the nonzero curvature case $m = 2$ (it is redundant if $m \ge 3$). The example is related to one by Oberlin and Smith [12], for a family of Bessel multipliers in $\mathbb{R}^2$ and for convolutions with arclength measure on a helix in $\mathbb{R}^3$; it is also closely related to examples for a deep inequality on decompositions of cone multipliers due to Wolff.

We describe Wolff's result [24] as it is crucial for deriving sufficient conditions for the L^p-Sobolev regularity of $\mathcal{R}$. One considers a collection $\{\Psi_\nu\}$ of smooth functions supported in disjoint $1 \times \delta^{1/2} \times \delta$-plates that are tangential to the light cone $\{\xi : \xi_3^2 = \xi_1^2 + \xi_2^2\}$ with the long side pointing in the radial direction; we assume that the Ψ_ν satisfy the natural size estimates and differentiability properties (see §3.1 below). Let f_ν be a family of tempered distributions. Wolff's theorem states that for sufficiently large q and for all $\varepsilon > 0$ there exists a finite $C_{\epsilon,q}$ such that

$$\Big\| \sum_\nu \widehat{\Psi}_\nu * f_\nu \Big\|_q \le C_{\epsilon,q} \delta^{-\frac{1}{2} + \frac{2}{q} - \epsilon} \Big(\sum_\nu \|f_\nu\|_q^q \Big)^{\frac{1}{q}}. \tag{1.3}$$

In [24] this inequality is established for all $q \geq 74$. In what follows we shall assume the validity of (1.3) for $q \geq q_W$. We show that the sharp $L^p \to L^p_{1-1/p}$ inequality holds for p close to 1; the precise range is determined by the value of q_W.

Theorem 1.2. *Suppose that $\gamma \in C^{m+3}(I)$ is of maximal type at most m, and suppose that $1 < p < \min\big((q_W+2)/q_W, (m+1)/m\big)$. Then $\mathcal{R}$ maps $L^p(\mathbb{R}^3)$ boundedly into $L^p_{1-1/p}(\mathbb{R}^3)$.*

This result is somewhat analogous to a recent result by the authors [16] on convolutions with measures supported on curves with nonvanishing curvature and torsion (see [13] for a prior partial result based on [1], [22]). A counterexample in [24] shows that necessarily $q_W \geq 6$ (*i.e.* (1.3) does not hold for $q < 6$). A proof of (1.3) for all $q > 6$ would imply the sharp endpoint $L^p \to L^p_{1-1/p}$ for all $1 < p < \min\{4/3, (m+1)/m\}$ and sharp results up to endpoints for larger p. In particular if $m > q_W/2$ (*i.e.* $m > 37$ according to [24]) then one obtains an almost complete result (except for endpoint bounds in the range $(m+1)/m \leq p < 2$).

2. Necessary conditions

We begin with two preliminary observations. Consider a multiplier $m(\xi')$ depending only on $\xi' = (\xi_1, \xi_2)$ and observe that $\mathcal{R}$ commutes with the operator $m(D')$. Now if $\mathcal{R}$ maps L^p to L^p_σ for some $p \in (1, \infty)$ and if m_k is a standard symbol of order zero in $\mathbb{R}^2$ which vanishes for $|\xi'| \leq 2^k$, then it follows

$$\|m_k(D')\mathcal{R}f\|_p \leq C_p 2^{-k\sigma}\|f\|_p. \tag{2.1}$$

Secondly, let $\mathcal{E}_k^\infty$ be the set of tempered distributions whose Fourier transform is supported in $\{\xi : |\xi| \geq 2^k\}$ (thus distributions in $\mathcal{E}_k^\infty$ have cancellation). Let Φ be smooth and supported in $\{\xi : |\xi| \leq 2\}$ and let $\Phi(\xi) = 1$ for $|\xi| \leq 1$. Define P_l by $\widehat{P_l f}(\xi) = \Phi(2^{-l}\xi)\widehat{f}(\xi)$. It is straightforward to see (using integration by parts arguments for generalized Fourier integrals with so called operator phase functions, see [10]) that there is the estimate

$$\|P_l\mathcal{R}f\|_p \leq C_N 2^{-Nk}\|f\|_p \quad \text{if } l < k - C_1, \quad f \in L^p \cap \mathcal{E}_k^\infty$$

and C_1 is large. This implies that if $\mathcal{R}: L^p \to L^p_\sigma$ then one also has for large k

$$\|\mathcal{R}f\|_p \lesssim 2^{-k\sigma}\|f\|_p, \quad f \in \mathcal{E}_k^\infty. \tag{2.2}$$

In what follows J will always denote a compact interval in $(1,2)$ so that $\chi_2(s) > 0$ on J. We shall also choose a fixed $a_0 \in I$ so that $\chi_1(a_0) \neq 0$.

2.1. $\sigma \le 1 - 1/p$.

Let η be an even Schwartz function in $\mathbb{R}^2$ with $\widehat{\eta}(0) = 1$ and with η supported in $\{\xi' : 1/2 \le |\xi'| \le 2\}$. Let c_0 so that $\widehat{\eta}(x') > 1/2$ for $|x'| \le c_0$. Let $m_k(\xi') = \eta(2^{-k}\xi')$, then

$$\begin{aligned} &m_k(D')\mathcal{R}f(x',\alpha) \\ &= (2\pi)^{-2}\chi_1(\alpha)\int 2^{2k}\widehat{\eta}(2^k(x'-y'+y_3\gamma(\alpha))f(y)\chi_2(y_3)dy. \end{aligned} \tag{2.3}$$

Now let s_0 be in te interior of J and let f_k be the characteristic function of a ball of radius $\varepsilon 2^{-k}$ centered at $(0,0,s_0)$. Then $\chi_2(s) \ge c' > 0$ for $|s - s_0| \le 2^{-k}$ if k is large. Let $\ell(\alpha)$ be the line segment $\{-s\gamma(\alpha) : s \in J\}$. For small ε let E_α be the set of all x' for which $\operatorname{dist}(x'; \ell(\alpha)) \le \varepsilon 2^{-k}$. As $\widehat{\eta}$ is positive near the origin we see that the integrand in (2.3) is $\ge c2^{2k}$ if $x' \in E_\alpha$. Thus the integral (2.3) is bounded below by 2^{-k}. Now E_α is of measure $\approx 2^{-2k}$ and after integrating in α we see that $\|m_k(D')\mathcal{R}f_k\|_p \gtrsim 2^{-k}2^{-2k/p}$. Since $\|f_k\|_p \lesssim 2^{-3k/p}$ we see that the L^p operator norm of $\mathcal{R}$ is at least $2^{-k(1-1/p)}$ and since η vanishes for $|\tau| \lesssim 2^k$ inequality (2.1) shows that the $L^p \to L^p_\sigma$ boundedness of $\mathcal{R}$ implies $\sigma \le 1 - 1/p$.

2.2. $\sigma \le (mp)^{-1}$.

Let us assume that γ has contact of order m with its tangent line at $\alpha = a_0$. Let ζ_1 be an even Schwartz function in $\mathbb{R}$ so that ζ_1 is supported in $\{\beta : 1/2 \le |\beta| \le 2\}$ and with the property that $\widehat{\zeta_1}(u) \ge 1/2$, $|u| \le c_0$. Let ζ_0 be a Schwartz function in $\mathbb{R}$ for which $\widehat{\zeta_0}$ is nonnegative everywhere and positive in $[-1/2, 1/2]$. Let η_k be defined by

$$\eta_k(\tau) = \zeta_0(2^{-k}\langle \tau, \gamma'(a_0)\rangle)\zeta_1(2^{-k}\langle \tau, \mathfrak{n}(a_0)\rangle)$$

where $\mathfrak{n} = (-\gamma_2', \gamma_1')$. The function η_k vanishes for $|\xi| \le c2^k$ and by (2.1) it suffices to prove that the L^p operator norm of $\eta_k(D')\mathcal{R}$ is $\gtrsim 2^{-k/(mp)}$.

Now let g_k be the characteristic function of the set defined by

$$|\langle y' - y_3\gamma(a_0), \gamma'(a_0)\rangle| \le 2^{-k/m}, |\langle y' - y_3\gamma(a_0), \mathfrak{n}(a_0)\rangle| \le 2^{-k}, y_3 \in J.$$

We evaluate $\eta_k(D')\mathcal{R}g_k(x', \alpha)$ on the set P_k defined by

$$|\langle x', \mathfrak{n}(a_0)\rangle| \le c\varepsilon 2^{-k}, |\langle x', \gamma'(a_0)\rangle| \le c\varepsilon 2^{-k/m}, |\alpha - a_0| \le c\varepsilon 2^{-k/m}.$$

Notice that if $y \in \operatorname{supp}(g_k)$ and $(x', \alpha) \in P_k$ then

$$|\langle x' - y' + \gamma(\alpha)y_3, \mathfrak{n}(a_0)\rangle| \ge c' 2^{-k}$$

since $\langle \gamma(\alpha) - \gamma(a_0), \mathfrak{n}'(a_0)\rangle = O(2^{-k})$ by the contact of order m assumption. Thus $\zeta_1(2^k\langle x' - y' + \gamma(\alpha)y_3, \mathfrak{n}(a_0)\rangle) \ge c'$ if $y \in \operatorname{supp}(g_k)$ and $(x', \alpha) \in P_k$. Because of the positivity assumption on ζ_0 we see that for $(x', \alpha) \in P_k$ and for fixed y_3, α the integral

$$\int \widehat{\eta}_k(x' - y' - \gamma(\alpha)y_3)g_k(y', y_3)dy'$$

is nonnegative, and if $y_3 \in J$ it is bounded below by a positive constant. Thus $|\eta_k(D')\mathcal{R}g_k(x', \alpha)| \ge c_1$ on P_k and therefore $\|\eta_k(D')\mathcal{R}g_k\|_p \ge c_2 2^{-k(m+2)/(mp)}$. Since $\|g_k\|_p \lesssim 2^{-k(m+1)/mp}$ we deduce our necessary condition $\sigma \le (mp)^{-1}$.

2.3. $\sigma \le 1/4$.

By a change of variable we may assume that γ is parametrized by arclength. We pick a closed interval I_0 in the support of χ_1 so that the curvature is bounded below in I_0, *i.e.*

$$|\kappa(\alpha)| = |\gamma_1''(\alpha)\gamma_2'(\alpha) - \gamma_1'(\alpha)\gamma_2''(\alpha)| \ge c > 0. \tag{2.4}$$

for $\alpha \in I_0$. Suppose that $|\gamma(\alpha)| \le B$ for all α. Let ρ be an even C^∞ function with the property that $\rho(x) \ge 1$ for $|x| \le 4B$ and so that $\widehat{\rho}$ is compactly supported.

We fix a positive integer n which will be chosen large (depending on the geometry). Let $k \gg 2n$. Let $\{\alpha_\nu\}$ be a maximal set of points in I_0 which have mutual distance $2^{n-k/2}$. Define

$$\xi_\nu = (\gamma_2'(\alpha_\nu), -\gamma_1'(\alpha_\nu), \gamma_1'(\alpha_\nu)\gamma_2(\alpha_\nu) - \gamma_1(\alpha_\nu)\gamma_2'(\alpha_\nu)) \tag{2.5}$$

and define

$$f_{k,\nu}(x) = \rho(x)e^{i2^k\langle \xi_\nu, x\rangle}$$

so that $\widehat{f_{k,\nu}}(\xi) = \widehat{\rho}(\xi - 2^k\xi_\nu)$. Let $\{r_\nu\}$ be the sequence of Rademacher functions and define for $\omega \in [0,1]$

$$f_k^\omega(x) = \sum_\nu r_\nu(\omega) f_{k,\nu}(x);$$

then by the usual L^p inequalities for the Rademacher functions [20]

$$\Big(\int_0^1 \|f_k^\omega\|_p^p d\omega\Big)^{1/p} \approx \Big\|\Big(\sum_\nu |f_{k,\nu}|^2\Big)^{1/2}\Big\|_p \approx 2^{k/4-n/2}\|\rho\|_p. \tag{2.6}$$

Observe that

$$\mathcal{R}f_{k,\mu}(x',\alpha) = \chi_1(\alpha)\int \rho(x'+s\gamma(\alpha),s)e^{i2^k\langle\xi'_\mu,x'+s\gamma(\alpha)\rangle+s\xi_{\mu3}}\chi_2(s)ds. \tag{2.7}$$

Now define $v_\nu^{(1)} = (\gamma(\alpha_\nu),1)$, $v_\nu^{(2)} = (\gamma'(\alpha_\nu),0)$ and check that both $\langle\xi_\nu, v_\nu^{(1)}\rangle = 0$, $\langle\xi_\nu, v_\nu^{(2)}\rangle = 0$. Moreover $\langle\xi'_\nu,\gamma''(\alpha_\nu)\rangle = \kappa(\alpha_\nu)$ and thus

$$\langle\xi'_\nu,\gamma(\alpha)\rangle + \xi_{\nu3} = \frac{\kappa}{2}(\alpha-\alpha_\nu)^2 + O(\alpha-\alpha_\nu)^3. \tag{2.8}$$

Observe that if $|x'|\le 1$ and $s\in J$ then $|x'+s\gamma(\alpha)|\le 4B$; moreover, by (2.8), we have $|e^{i\langle\xi'_\nu,\gamma(\alpha)\rangle+\xi_{\nu3}}-1|\le 1/2$ if $|\alpha-\alpha_\nu|\le c2^{-k/2}$ and c is small. By (2.7),

$$\text{Re}\,\big(Rf_{k,\nu}(x',\alpha)\big) \ge c_1 \quad \text{if } |\alpha-\alpha_\nu|\le c2^{-k/2}, |x'|\le 1$$

and consequently

$$\Big(\sum_\nu \int_{|\alpha-\alpha_\nu|\le 2^{-k/2}}\int_{|x'|\le 1} |\mathcal{R}f_{k,\nu}(x',\alpha)|^p d\alpha dx'\Big)^{1/p} \ge c_2 2^{-n/p}. \tag{2.9}$$

Now we find an upper bound for $|\mathcal{R}f_{k,\mu}(x',\alpha)|$ when $|\alpha-\alpha_\nu|\lesssim 2^{-k/2}$ and $\mu\ne\nu$. We use (2.7), (2.8) and apply integration by parts to see that

$$|\mathcal{R}f_{k,\mu}(x',\alpha)| \le C_N(2^k|\alpha_\nu-\alpha_\mu|^2)^{-N}, \text{ if } |\alpha-\alpha_\nu|\le c2^{-k/2},\ \mu\ne\nu.$$

Thus by the separation property of the α_μ

$$\Big(\sum_\nu \int_{|\alpha-\alpha_\nu|\le 2^{-k/2}}\int_{|x'|\le 1} \big[\sum_{\mu\ne\nu}|\mathcal{R}f_{k,\mu}(x',\alpha)|\big]^p d\alpha dx'\Big)^{1/p} \le C_N 2^{-nN}. \tag{2.10}$$

If n is chosen sufficiently large a combination of (2.9) and (2.10) yields that $\|\mathcal{R}f_k^\omega\|_p \ge c(p) > 0$ uniformly in ω. Since $\widehat{\rho}$ has compact support and $|\xi_\nu| \geq 1$ the Fourier transforms of the functions f_k^ω are supported in $\{\xi : |\xi| \geq c_3 2^k\}$ if k is sufficiently large. Using (2.6) we see that for large k

$$\sup\{\|\mathcal{R}f\|_p : \|f\|_p \leq 1, f \in \mathcal{E}_k^\infty \cap L^p\} \gtrsim 2^{-k/4}$$

and thus by the consideration leading to (2.2) the operator $\mathcal{R}$ does not map L^p to L^p_σ if $\sigma > 1/4$.

3. L^p regularity

3.1. **Preliminaries.** We begin by describing an extension of Wolff's inequality proved in [16]. Let $\alpha \mapsto g(\alpha) = (g_1(\alpha), g_2(\alpha)) \in \mathbb{R}^2$ be a C^3 curve on the plane defined on a closed subinterval I of $[-1, 1]$. We assume that for positive constants b_0, b_1, b_2,

$$\|g\|_{C^3(I)} \leq b_0, \quad |g'(\alpha)| \geq b_1, \quad |g_1'(\alpha)g_2''(\alpha) - g_2'(\alpha)g_1''(\alpha)| \geq b_2. \tag{3.1}$$

Given $\alpha \in I$, we define three vectors

$$u_1(\alpha) = (g(\alpha), 1), \quad u_2(\alpha) = (g'(\alpha), 0), \quad u_3(\alpha) = u_1(\alpha) \times u_2(\alpha), \tag{3.2}$$

so that a basis of the tangent space of the cone $\mathcal{C}_g = \{rg(\alpha)\}$ is given by $\{u_1(\alpha), u_2(\alpha)\}$. Then for given $\lambda > 0$ and $0 < \delta \ll 1$, the (δ, λ)-*plate* at α, denoted by $P^\alpha_{\delta,\lambda}$ is defined to be the parallelepiped

$$P^\alpha_{\delta,\lambda} = \{\xi : \lambda/2 \leq |\langle u_1(\alpha), \xi\rangle| \leq 2\lambda, \ |\langle u_2(\alpha), \xi - \xi_3 u_1(\alpha)\rangle| \leq \lambda\delta^{1/2}, \\ |\langle u_3(\alpha), \xi\rangle| \leq \lambda\delta\}.$$

Note that $P^\alpha_{\delta,\lambda}$ has dimension $\approx \lambda$ in the radial direction tangent to the cone $\mathcal{C}_g$, dimension $\approx \lambda\delta^{1/2}$ in the tangential direction perpendicular to the radial direction, and is supported in a neighborhood of width $\approx \lambda\delta$ of the cone. An A-*extension of the plate* $P^\alpha_{\delta,\lambda}$ is a parallelepiped that is localized between heights $\xi_3 = \lambda/(2A)$ and $\xi_3 = 2A\lambda$ of $\mathcal{C}_g$, and whose width along $(u_2(\alpha), -\langle u_1(\alpha), u_2(\alpha)\rangle)$ and $u_3(\alpha)$ are $A\lambda\delta^{1/2}$ and $A\lambda\delta$ respectively. For θ and σ with $\sigma \leq \delta^{1/2} \leq \theta$, a $(\delta, \lambda, \theta)$-*plate family* associated to g is a finite collection of (δ, λ)-plates $\mathcal{P} = \{P^{\alpha_\nu}_{\delta,\lambda}\}_{\nu=1}^N$ satisfying (i) $|\alpha_\nu - \alpha_{\nu'}| \geq \delta^{1/2}$ and (ii) $\max_\nu\{\alpha_\nu\} - \min_\nu\{\alpha_\nu\} \leq \theta$. An *admissible bump function* associated to $P^\alpha_{\delta,\lambda}$ is a C^∞ function ϕ

supported in $P^\alpha_{\delta,\lambda}$ satisfying the estimates

$$|\langle u_1(\alpha),\nabla\rangle^{n_1}\langle u_2(\alpha),\nabla\rangle^{n_2}\langle u_3(\alpha),\nabla\rangle^{n_3}\phi(\xi)| \le \lambda^{-n_1-n_2-n_3}\delta^{-n_2/2-n_3}, \quad 0\le n_1+n_2+n_3\le 4. \tag{3.3}$$

The Wolff inequality in this general context says that

$$\Big\|\sum_{P\in\mathcal{P}}\mathcal{F}^{-1}[\phi_P\widehat{f}_P]\Big\|_q \le C(\epsilon)\delta^{\frac{2}{q}-\frac12-\epsilon}\Big(\sum_{P\in\mathcal{P}}\|f_P\|_q^q\Big)^{1/q}, \tag{3.4}$$

where $\{\phi_P\}$ is a collection of admissible bump function associated to the plates in $\mathcal{P}$. Wolff [24] proved this for the light cone, *i.e.* $g(\alpha)=(\cos\alpha,\sin\alpha)$ (when $q>74$) but the authors showed in §2 of [16] that if (3.4) holds for the light-cone and some q then it holds for the same q for every curved cone generated by g as in (3.1) (with a different constant $\widetilde{A}(\varepsilon)$). The proof involves various rescaling and an induction on scales argument.

We now describe the structure of the wavefront set of the Schwartz kernel of the operator $\mathcal{R}$. We shall assume that our curve γ is parametrized by arclength. Moreover we deal with the case $m=2$ of Theorem 1.2 and assume the lower bound (2.4) for the curvature everywhere in $\operatorname{supp}(\chi_1)$. It will be convenient to work with the adjoint operator $\mathcal{R}^*$, and we write out the convolution kernel by expanding a Dirac measure in two dimensions by a Fourier integral; thus

$$\begin{aligned}\mathcal{R}^*f(x) &= \chi_2(x_3)\int f(x'-x_3\gamma(y_3),y_3)\,\chi_1(y_3)\,dy_3\\ &= \chi_2(x_3)\int e^{i\varphi(x,y,\tau)}\chi_1(y_3)\,f(y)\,d\tau\,dy,\end{aligned} \tag{3.5}$$

where $x=(x',x_3)$, $y=(y',y_3)$, $\tau=(\tau_1,\tau_2)$, and

$$\varphi(x,y,\tau)=\sum_{i=1}^2\tau_i(y_i-x_i+x_3\gamma(y_3)). \tag{3.6}$$

The theorem will be proved if we can show that $\mathcal{R}^*$ maps $L^q_{-1/q}$ to L^q (or more generally $L^q_{\beta-1/q}$ to L^q_β for all β), for $q>(q_W+2)/2$.

We record a few standard facts about $\mathcal{R}^*$ that will be used in the analysis. We denote (x,ξ)-space by $T^*\mathbb{R}^3_L$ and (y,η)-space by $T^*\mathbb{R}^3_R$ and the canonical relation associated to $\mathcal{R}^*$ is given by

$$\begin{aligned}\mathfrak{C} &= \{(x, \varphi_x, y, -\varphi_y) : \varphi_\tau = 0\} \\ &= \big\{(x, \xi, y, -\eta) \,: \xi' = -\tau,\, \xi_3 = \langle \tau,\, \gamma(y_3)\rangle,\, y' = x' - x_3\gamma'(y_3), \\ &\qquad\qquad\qquad\qquad\qquad \eta' = \tau,\, \eta_3 = x_3\langle \tau, \gamma'(y_3)\rangle\big\}.\end{aligned}$$

Let π_L, π_R be the projections of $\mathfrak{C}$ to $T^*\mathbb{R}^3_L$ and $T^*\mathbb{R}^3_R$, respectively. The structure of the projections π_L, π_R for more general X-ray transform satisfying a version of the Gelfand admissibility condition has been investigated in [6], [7], [9] (see also the survey [15]), namely π_L is a fold and π_R is a blowdown. In particular the $L^2_a \to L^2_{a+1/4}$ estimates are shown in these references (and in our model case this result is rather straightforward as $\mathcal{R}\mathcal{R}^*$ is a convolution operator).

Let $\mathfrak{C}^{\mathrm{deg}}$ be the variety where $\det \pi_L = 0$ (equivalently $\det d\pi_R = 0$); then $\mathfrak{C}^{\mathrm{deg}}$ is a conic submanifold of $\mathfrak{C}$ and the restriction of π_L to $\mathfrak{C}^{\mathrm{deg}}$ is locally a diffeomorphism onto a conic hypersurface of $T^*\mathbb{R}^3_L$. Moreover the sets $\Sigma_x = \{\xi : (x, \xi) \in \pi_L\mathfrak{C}^{\mathrm{deg}}\}$ are smooth two-dimensional cones in each fiber. In our special case the condition $\det d\pi_{L/R} = 0$ reduces to

$$\langle \tau, \gamma'(y_3)\rangle = 0 \tag{3.7}$$

and the cones Σ_x are given by

$$\Sigma = \big\{\xi \in \mathbb{R}^3 \,:\, \xi = \lambda(\gamma_2', -\gamma_1', -\gamma_1\gamma_2' + \gamma_1'\gamma_2),\, \lambda \in \mathbb{R}\big\}\,.$$

Recall that for our example in §2.3 the points ξ_ν from (2.5) were chosen to lie on Σ.

A simple computation shows that the cone Σ has one principal non-vanishing curvature. Indeed after suitable localization and rotation we may assume $|\gamma_1'(y_3)| > 1/2$ for all $y_3 \in I$. Then

$$g(\alpha) = \Big(-\frac{\gamma_2'(\alpha)}{\gamma_1'(\alpha)}, \frac{\gamma_1(\alpha)\gamma_2'(\alpha)}{\gamma_1'(\alpha)} - \gamma_2(\alpha)\Big) \tag{3.8}$$

parametrizes the curve that is the cross-section $\xi_2 = 1$ of Σ, and the curvature property of Σ can be expressed in terms of the curvature of g. A computation shows that

$$\det\begin{pmatrix} g_1'(\alpha) & g_2'(\alpha) \\ g_1''(\alpha) & g_2''(\alpha)\end{pmatrix} = -\frac{\kappa^2(\alpha)}{(\gamma_1'(\alpha))^3} \neq 0.$$

Thus g satisfies all the conditions of (3.1), with b_0, b_1 and b_2 depending only on $\|\gamma\|_{C^2}$ and lower bounds of $|\kappa|$. We also observe that Σ is the cone dual to $\mathcal{C}_\gamma$; indeed a normal vector to Σ at $r(\gamma_2', -\gamma_1', \gamma_1'\gamma_2 - \gamma_1\gamma_2') = (\gamma', 0) \times (\gamma, 1)$ is given by $(\gamma(\alpha), 1)$.

3.2. **Dyadic estimates.** We first decompose the oscillatory integral (3.5) dyadically in τ and then we introduce a further decomposition in terms of the size of $|\det \pi_L| \approx |\langle \tau, \gamma'(y_3)\rangle|$. In what follows $m_k(\tau)$ will be a standard multiplier symbol of order 0 supported where $|\tau| \approx 2^k$, and

$$R_k f(x) = \chi_2(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3)\, m_k(\tau) d\tau\, f(y)\, dy,$$

where φ is as in (3.6).

We shall prove here (under the assumption (2.4)) that

$$\|R_k f\|_q \le C_q 2^{-k/q} \|f\|_q, \quad q > (q_W + 2)/2, \tag{3.9}$$

which implies an estimate for $\mathcal{R}^*$ on Besov spaces. The Sobolev estimates will be briefly discussed in §3.3.

To describe our further decomposition let $\eta_0 \in C_0^\infty(\mathbb{R})$ be an even function so that $\eta_0(s) = 1$ if $|s| \le 1/2$ and $\text{supp}(\eta_0) \subset (-1, 1)$, and let $\eta_1(s) = \eta_0(s/2) - \eta_0(s)$. Define

$$a_{k,l}(y_3, \tau) = m_k(\tau) \eta_1(2^{l-k} \langle \gamma'(y_3), \tau \rangle) \tag{3.10}$$

and

$$b_k(y_3, \tau) = m_k(\tau) \big(1 - \sum_{l < k/2} \eta_1(2^{l-k} \langle \gamma'(y_3), \tau \rangle)\big).$$

Let

$$R_{k,l} f(x) = \chi_2(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3)\, a_{k,l}(\tau) d\tau\, f(y)\, dy,$$

and define $\widetilde{R}_k$ similarly, with $a_{k,l}$ replaced by b_k.

Proposition 3.1. *For $q_W < q < \infty$,*

$$\|R_{k,l} f\|_q \le C_\epsilon 2^{-k/q} 2^{-l/q + l\epsilon} \|f\|_q, \quad l < k/2, \tag{3.11}$$

$$\|\widetilde{R}_k f\|_q \le C_\epsilon 2^{-3k/(2q) + k\epsilon} \|f\|_q. \tag{3.12}$$

The constants C_ϵ depend only on ϵ, $\|\gamma\|_{C^2}$ and the lower bound in (2.4).

Proof. We give the proof of (3.11). The proof of (3.12) is similar, with mainly notational changes. We note that $R_{k,l} = 0$ for $l < -C$ and that the asserted bound for small l follows from standard estimates for generalized Radon transforms or Fourier integral operators ([10]).

By a calculation with Fourier transforms we get

(3.13)

$$\mathcal{F}_{\mathbb{R}^3}\big[R_{k,l}f\big](\xi', \xi_3) = \int \widehat{\chi}_2(\xi_3 + \langle\gamma(y_3), \xi'\rangle) a_{k,l}(y_3, \xi')\mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3) dy_3$$

In view of the fast decay of $\widehat{\chi}_2$ we further split χ_2 in a low and a high frequency part. Define $\vartheta_{k,l}$ by

$$\widehat{\vartheta_{k,l}}(\beta) = \eta_0(2^{-k+2l(1-\epsilon)}\beta)\widehat{\chi}_2(\beta)$$

and split

$$R_{k,l} = T_{k,l} + E_{k,l} \tag{3.14}$$

where $T_{k,l}$ is similarly defined as $R_{k,l}$ but with $\chi_2(x_3)$ replaced by $\vartheta_{k,l}(x_3)$.

The error term $E_{k,l}$ is easily handled; we claim that given any $q \geq 2$ and $N \geq 1$, there exists $C_{q,N} > 0$ such that

$$\|E_{k,l}f\|_q \leq C_{q,N} 2^{-(k-2l+2l\epsilon)N/q} \|f\|_q. \tag{3.15}$$

Indeed by an integration by parts it is easy to see that $E_{k,l}$ is bounded on L^∞. Thus, by interpolation it suffices to consider the case $q = 2$. We use the formula analogous to (3.13), with $\chi_2 - \vartheta_{k,l}$ in place of χ_2, and the fact that the Fourier transform of $\chi_2 - \vartheta_{k,l}$ vanishes for $|\beta| > c2^{k-2l+2l\epsilon}$. Consequently, by the Cauchy-Schwarz inequality and Plancherel's theorem

$$\begin{aligned} \|E_{k,l}f\|_2^2 &\leq C_N \int \int_{\substack{|\xi_3 + \langle\gamma(y_3),\xi'\rangle| \\ \gtrsim 2^{k-2l+2l\epsilon}}} \frac{|\mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3)|^2}{|\xi_3 + \langle\gamma(y_3), \xi'\rangle|^{2N}} \, d\xi dy_3 \\ &\leq C_N 2^{-l} 2^{-(k-2l+2l\epsilon)(2N-1)} \|f\|_2^2 \end{aligned}$$

which is (3.15)

In order to estimate the main term $T_{k,l}$ in (3.14) we further decompose $a_{k,l}$ into pieces supported on 2^{-l} subintervals of I. Let $\zeta \in C_0^\infty$ be supported in $(-1, 1)$ so that $\sum_{\nu\in\mathbb{Z}} \zeta(\cdot - \nu) \equiv 1$. We set

$$a_{k,l,\nu}(y_3, \tau) = \zeta(2^l y_3 - \nu) a_{k,l}(y_3, \tau)$$

so that $a_{k,l} \equiv \sum_\nu a_{k,l,\nu}$, and let

$$T_{k,l,\nu} f(x) = \vartheta_{k,l}(x_3) \int e^{i\varphi(x,y,\tau)} \chi_1(y_3)\, a_{k,l,\nu}(\tau) d\tau\, f(y)\, dy,$$

Notice the natural estimates for the derivatives of $a_{k,l,\nu}$, namely if $|\alpha - 2^{-l}\nu| \le 2^{-l}$ and if $\mathfrak{n}(\alpha) = (-\gamma_2'(\alpha), \gamma_1'(\alpha))$ then

$$\big|\langle \gamma'(\alpha), \nabla_\tau\rangle^{n_1} \langle \mathfrak{n}(\alpha), \nabla_\tau\rangle^{n_2} a_{k,l,\nu}(y_3,\tau)\big| \le C_{n_1,n_2} 2^{(l-k)n_1} 2^{-kn_2}. \tag{3.16}$$

Moreover, the Fourier transforms of the functions $T_{k,l,\nu} f$ have support properties which are favorable for the application of Wolff's inequality. Let $\alpha_\nu = 2^{-l}\nu$, and let $\{u_\nu^{(1)}, u_\nu^{(2)}, u_\nu^{(3)}\}$ be an orthonormal basis where $u_\nu^{(3)}$ is parallel to $(\gamma(\alpha_\nu), 1)$, $u_\nu^{(1)}$ is orthogonal to both $(\gamma(\alpha_\nu), 1)$ and $(\gamma'(\alpha_\nu), 0)$. Thus if $\xi = r(\gamma_2', -\gamma_1', \gamma_1'\gamma_2 - \gamma_1\gamma_2')(\alpha_\nu)$ then $u_\nu^{(3)}$ is normal to Σ at ξ and $u_\nu^{(1)}$, $u_\nu^{(2)}$ are tangent vectors. One now verifies that the Fourier transform of $T_{k,l,\nu} f$ is supported in a set where

$$C^{-1} \le |\langle \tfrac{\xi}{|\xi|}, u_\nu^{(1)}\rangle| \le C, \tag{3.17a}$$

$$|\langle \tfrac{\xi}{|\xi|}, u_\nu^{(2)}\rangle| \le C 2^{-l}, \tag{3.17b}$$

$$|\langle \tfrac{\xi}{|\xi|}, u_\nu^{(3)}\rangle| \le C 2^{-2l+2l\epsilon}. \tag{3.17c}$$

Indeed we have

$$\begin{aligned} &\mathcal{F}_{\mathbb{R}^3}\big[T_{k,l,\nu} f\big](\xi', \xi_3) \\ &\quad = \int \widehat{\vartheta_{k,l}}(\xi_3 + \langle \gamma(y_3), \xi'\rangle) a_{k,l,\nu}(y_3, \xi') \mathcal{F}_{\mathbb{R}^2} f(-\xi', y_3) dy_3. \end{aligned} \tag{3.18}$$

To see the assertion on the support we first note that (3.17a) follows since our symbols are supported near Σ and away from the origin. Next $|\langle \xi, u_\nu^{(3)}\rangle| \approx |\xi_3 + \langle \gamma(\alpha_\nu), \xi'\rangle|$ so that (3.17c) is an immediate consequence of the support property of $\widehat{\vartheta_{k,l}}$ and the fact that $\alpha_\nu \in \text{supp } a_{k,l,\nu}(\cdot, \xi')$. To show (3.17b) let $\mathfrak{t}_\nu = (\gamma'(\alpha_\nu), 0))$ and observe that $u_\nu^{(2)}$ belongs to the span of $\mathfrak{t}_\nu$ and $u_\nu^{(3)}$. By the definition of $a_{k,l,\nu}$ we have $\langle \mathfrak{t}_\nu, \xi\rangle = O(2^{k-l})$ and this together with (3.17c) implies (3.17b).

By (3.17), the Fourier transforms of $\widehat{T_{k,l,\nu} f}$ are supported in C-extensions of plates P_ν which form a family of $(2^k, 2^{-2l+2l\epsilon})$ plates generated by g as in (3.8). After a straightforward reduction we may apply our variant of Wolff's inequality (3.4) and get for $q > q_W$

$$\Big\| \sum_\nu T_{k,l,\nu} f \Big\|_q \le C_\epsilon 2^{2l\left(\frac12 - \frac2q + \epsilon\right)} \Big(\sum_\nu \|T_{k,l,\nu} f\|_q^q \Big)^{1/q}. \tag{3.19}$$

In order to finish the proof we need to prove that

$$\|T_{k,l,\nu}\|_{L^2\to L^2} \le C 2^{(l-k)/2}, \tag{3.20}$$

$$\|T_{k,l,\nu}\|_{L^\infty\to L^\infty} \le C 2^{-l}. \tag{3.21}$$

These inequalities imply the stronger bound

$$\Big(\sum_\nu \|T_{k,l,\nu} f\|_q^q \Big)^{1/q} \lesssim 2^{-l(1-3/q)} 2^{-k/q} \|f\|_q. \tag{3.22}$$

For $q = 2$ (3.22) follows from (3.20) and the almost disjointness of the supports of $y_3 \mapsto a_{k,l,\nu}(\tau, y_3)$. For $2 \le q < \infty$ it follows by interpolation with the $\ell^\infty(L^\infty)$ bound from (3.21). The asserted estimate for $R_{k,l}$ follows then by combining (3.19) and (3.22).

We conclude by proving (3.20) and (3.21). To see (3.20) we recall that $|\langle \gamma'(y_3), \tau\rangle| \approx 2^{k-l}$ on the support of $a_{k,l,\nu}$ and write $a_{k,l,\nu} = a^+_{k,l,\nu} + a^-_{k,l,\nu}$, where $a^\pm_{k,l,\nu}$ further localize to the regions where $\langle \gamma'(y_3), \tau\rangle$ is positive and negative, respectively. Consequently we get a decomposition $T_{k,l,\nu} = T^+_{k,l,\nu} + T^-_{k,l,\nu}$. We only work with $\mathcal{T} \equiv T^+_{k,l,\nu}$, the other case being similar. Let

$$\begin{aligned} K(x,\tau,y_3) &= \vartheta_{k,l}(x_3) \int e^{i[\langle x-z,\xi\rangle + \langle \tau, -z' + z_3\gamma(y_3)\rangle]} a_{k,l,\nu}(y_3,\xi')\, dz\, d\xi, \\ &= e^{-i\langle \tau, x' - x_3\gamma(y_3)\rangle} \vartheta_{k,l}(x_3) a_{k,l,\nu}(y_3, -\tau); \end{aligned}$$

then $\mathcal{T} f(x) = \int K(x, y_3, \tau) \mathcal{F}_{\mathbb{R}^2} f(\tau, y_3) d\tau dy_3$ and by Schur's lemma and Plancherel's theorem

$$\begin{aligned} \|\mathcal{T}\|^2_{L^2\to L^2} &\le \sup_{\tau, y_3} \iint \Big| \int K(x,\tau,y_3) \overline{K(x,\zeta',z_3)} dx \Big| d\zeta' dz_3 \\ &\lesssim \|a^+_{k,l,\nu}\|_\infty \int \Big| \int e^{ix_3\langle \gamma(y_3) - \gamma(z_3), \tau\rangle} |\vartheta_{k,l}(x_3)|^2 \overline{a^+_{k,l,\nu}(z_3, -\tau)} dx_3 \Big| dz_3 \end{aligned}$$

which is controlled by

$$\sup_\tau \int \frac{|a^+_{k,l,\nu}(z_3,-\tau)|}{(1+|\langle\gamma(y_3)-\gamma(z_3),\tau\rangle|)^2}dz_3$$

$$\le C\sup_{y_3}\int \frac{dz_3}{(1+2^{k-l}|y_3-z_3|)^2} \lesssim 2^{l-k}. \tag{3.23}$$

Here we integrated by parts twice in x_3 and for the last estimate we used $|\langle\gamma(y_3)-\gamma(z_3),\tau\rangle| = |y_3-z_3||\langle\gamma'(w),\tau\rangle| \approx 2^{k-l}|y_3-z_3|$; the point w lies between y_3 and z_3, and since we work with $a^+_{k,l,\nu}$ the quantity $\langle\gamma'(\cdot),\tau\rangle$ does not change sign in $[y_3,z_3]$. (3.23) yields the L^2 bound (3.20).

For the L^∞ estimate, we integrate by parts in τ using the directional derivatives $\langle\gamma'(\alpha_\nu),\nabla_\tau\rangle$ and $\langle\mathfrak{n}(\alpha_\nu),\nabla_\tau\rangle$ and the symbol estimates (3.16). This gives

$$\|\mathcal{T}f\|_\infty \le \sup_x\Big|\int\Big[\int e^{i\langle\tau,y'-x'+x_3\gamma(y_3)\rangle}\vartheta_{k,l}(x_3)a(y_3,-\tau)\,d\tau\Big]f(y)\,dy\Big|$$

$$\lesssim \|f\|_\infty \sup_x\int_{\text{supp}(a)}(1+2^k|\langle\mathfrak{n}(\alpha_\nu),y'-x'+x_3\gamma(y_3)\rangle|)^{-2}\times$$

$$(1+2^{k-l}|\langle\gamma'(\alpha_\nu),y'-x'+x_3\gamma(y_3)|)^{-2}\,d\tau dy$$

and we integrate first in y' and then in y_3,τ over the set where

$$|\langle\gamma'(\alpha_\nu),\tau\rangle|\lesssim 2^{k-l},\quad |\langle\mathfrak{n}(\alpha_\nu),\tau\rangle|\lesssim 2^k,\quad |y_3-\alpha_\nu|\lesssim 2^{-l}.$$

The result is the asserted bound $\|\mathcal{T}f\|_\infty \lesssim 2^{-l}\|f\|_\infty$. This concludes the proof of the proposition. ☐

The proposition, and a further interpolation yield

Corollary 3.2. *For $q>(q_W+2)/2$, there is $\epsilon_0=\epsilon_0(q)>0$ such that*

$$\|R_{k,l}f\|_q \le C_q 2^{-k/q}2^{-\epsilon_0 l/q}\|f\|_q. \tag{3.24}$$

Proof. By the almost disjointness of the plate families and the L^2 bounds in (3.20) we see that the L^2 operator norm of $T_{k,l}$ is $O(2^{(l-k)/2})$, and by (3.15) this also holds for $R_{k,l}$. Interpolating this with the L^q bounds of Proposition 3.1 yields the assertion. ☐

3.3. **Bounds in Sobolev spaces.** We now complete the proof of Theorem 1.2 in the nonvanishing curvature case; here we still have to put the estimates (3.24) for different k together. We wish to show that $\mathcal{R}^*$ maps L^q_β to $L^q_{\beta+1/q}$ for $q > (q_W+2)/2$, and by duality (with $\beta = -1/q$) this will imply the asserted $L^p \to L^p_{1-1/p}$ for $\mathcal{R}$. By a vector-valued version of the Fefferman-Stein inequality for the #-function, Littlewood-Paley theory and standard integration by parts arguments (as in [10] or [18], p. 695) one reduces matters to an estimate for

(3.25)
$$S_l F(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q \Big(\sum_{k\ge 2l} \Big|2^{k/q}\big[R_{k,l}f_k(w) - \frac{1}{|Q|}\int R_{k,l}f_k(z)dz\big]\Big|^2\Big)^{1/2}$$

where $F = \{f_k\}_{k\in\mathbb{N}}$ and the supremum is taken over all cubes Q containing x. The assertion follows from

$$\|S_l F\|_q \le 2^{-l\varepsilon(q)}\|F\|_{L^q(\ell^2)}, \quad \frac{q_W+2}{2} < q < \infty. \tag{3.26}$$

One splits $S_lF(x)$ into three parts. The main part is concerned with the terms where $|2^k \mathrm{diam}(Q)| \le 2^{Cl}$ for some large C; here one applies Hölder's inequality in k and uses the dyadic L^q estimates above (*cf.* Corollary 3.2). The terms with $|2^k\mathrm{diam}(Q)| > 2^{Cl}$ are dealt with by standard L^2 and $L^\infty \to BMO$ bounds for generalized Radon transforms. We omit the details which are very similar to those in §3 of [16] (based on arguments in [17] in a different context).

3.4. **Extension to curves of finite type.** We now relax the curvature assumption on γ and assume that γ is of finite maximal type $\le m$. In the terminology of [18] the operator $\mathcal{R}$ satisfies a *right finite type condition* of order $\le m+1$, while in the terminology of [19] the underlying incidence relation is of type $\preceq (m,1)$. Finally in Comech's terminology [2] the projection π_R (for the canonical relation associated to $\mathcal{R}$) is of type $\le m-1$ (while π_L as a blowdown is not of finite type).

We may relax the finite type assumption a bit by not necessarily assuming that $\gamma' \ne 0$. We shall fix $a_0 \in I$ and estimate R under the assumption that χ_1 is supported in a small neighborhood of α_0. Assume

$$\gamma(a_0+\alpha) = \gamma(a_0) + (\beta_1\alpha^{n_1}\varphi_1(\alpha), \beta_2\alpha^{n_2}\varphi_2(\alpha)),$$

where $1 \le n_1 < n_2 \le m$, β_1 and β_2 are nonzero constants and $\varphi_i \in C^{m+5-n_i}$ with $\varphi_i(0) = 1$. We may reduce Theorem 1.2 to this case (with $n_1 = 1$) by localization and perhaps a rotation. Furthermore, we may assume that $1/2 \le \varphi_i \le 3/2$, $i = 1, 2$ on $\mathrm{supp}(\chi_1)$.

We work with a dyadic partition of unity $\zeta_j(\alpha) = \zeta(2^j(\alpha - a_0))$ where ζ_j is supported in the two intervals where $|\alpha| \approx 2^{-j}$. Let $\mathcal{R}_j f(x', \alpha) = \zeta_j(\alpha)\mathcal{R}f(x', a)$. We claim that for $1 < p < (q_W + 2)/q_W$

$$\big\|\mathcal{R}_j f\|_{L^p_{1-1/p}} \le C 2^{j(m - \frac{m+1}{p})} \|f\|_p; \tag{3.27}$$

this clearly yields the assertion of the theorem. We use a simple scaling argument. For $|u| \approx 1$ define

$$\Gamma_j(u) = \big(\beta_1 u^{n_1}\phi_1(a_0 + 2^{-j}u), \beta_2 u^{n_2}\phi_2(a_0 + 2^{-j}u)\big);$$

and let

$$T_j f(x', u) = \zeta(u) \int \chi_2(s) f(x' + s\Gamma_j(u), s) ds.$$

Notice that $\det(\Gamma_j'(u), \Gamma_j''(u) \approx \beta_1\beta_2 u^{n_1+n_2}(1 + O(2^{-j}))$ and that the derivatives of Γ_j are uniformly bounded above. Thus by our estimate for the nonvanishing curvature case the operators T_j map L^p to $L^p_{1-1/p}$, $1 < p < (q_W + 2)/q_W$, with bounds uniform in j. A short computation shows that $\mathcal{R}_j f(2^{-jn_1}x_1, 2^{-jn_2}x_2, a_0 + 2^{-j}u) = \chi_1(a_0 + 2^{-j}u) T_j f_j(x', u)$ where $f_j(y) = f(2^{-jn_1}y_1, 2^{-jn_2}y_2, y_3)$. Since $\max\{n_1, n_2\} \le m$ the inequality (3.27) follows quickly. □

References

[1] J. Bourgain, *Estimates for cone multipliers,* Geometric Aspects of Functional Analysis, Operator theory, Advances and Applications, vol. 77, ed. by J. Lindenstrauss and V. Milman, Birkhäuser Verlag, 1995.

[2] A. Comech, *Optimal regularity of Fourier integral operators with one-sided folds,* Comm. Partial Differential Equations, 24 (1999), 1263–1281.

[3] I.M.Gelfand and M.I. Graev, *Line complexes in the space* $\mathbb{C}^n$*,* Func. Ann. Appl., 2 (1968) 219-229.

[4] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities,* J. reine ang. Math., 455, (1994), 35–56.

[5] ———, *Oscillatory and Fourier integral operators with degenerate canonical relations,* Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. 2002, Vol. Extra, 93–141.

[6] A. Greenleaf and G. Uhlmann, *Nonlocal inversion formulas for the X-ray transform,* Duke Math. J., 58 (1989), 205–240

[7] ———, *Estimates for singular Radon transforms and pseudo-differential operators with singular symbols,* J. Funct. Anal., 89 (1990) 202–232.

[8] ———, *Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms, I,* Ann. Inst. Fourier (Grenoble) 40 (1990), no. 2, 443–466.

[9] V. Guillemin, Cosmology in (2 + 1)-dimensions, cyclic models, and deformations of $M_{2,1}$, Annals of Mathematics Studies, 121, Princeton University Press, Princeton, NJ, 1989.

[10] L. Hörmander, *Fourier integral operators, I,* Acta Math. 127 (1971), 79–183.

[11] N. Laghi, *Elements of functional calculus and L^2 regularity for some classes of Fourier integral operators*, preprint.

[12] D. Oberlin and H. Smith, *A Bessel function multiplier,* Proc. Amer. Math. Soc. 127 (1999), 2911–2915.

[13] D. Oberlin, H. Smith, and C. Sogge, *Averages over curves with torsion,* Math. Res. Lett. 1998, 535–539.

[14] D.H. Phong, E.M. Stein, *Radon transforms and torsion,* Internat. Math. Res. Notices 1991, no. 4, 49–60.

[15] D.H. Phong, *Singular integrals and Fourier integral operators,* Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 286–320, Princeton Math. Ser., 42, Princeton Univ. Press, Princeton, NJ, 1995.

[16] M. Pramanik and A. Seeger, *L^p regularity of averages over curves and bounds for associated maximal operators,* preprint.

[17] A. Seeger, *Some inequalities for singular convolution operators in L^p-spaces,* Trans. Amer. Math. Soc., 308 (1988) 259–272.

[18] ———, *Degenerate Fourier integral operators in the plane,* Duke Math. J. 71 (1993), no. 3, 685–745.

[19] ———, *Radon transforms and finite type conditions,* J. Amer. Math. Soc. 11 (1998), no. 4, 869–897.

[20] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, N.J., 1971.

[21] ———, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton University Press, Princeton, NJ, 1993.

[22] T. Tao and A. Vargas, *A bilinear approach to cone multipliers*, *I. Restriction estimates*, Geom. Funct. Anal. 10 (2000), 185–215; *II, Applications*, Geom. Funct. Anal. 10 (2000), 216–258.

[23] T. Wolff, *A sharp bilinear cone restriction estimate,* Ann. of Math., 153 (2001), 661–698.

[24] ———, *Local smoothing type estimates on L^p for large p,* Geom. Funct. Anal. 10 (2000), no. 5, 1237–1288.

M. Pramanik, Department of Mathematics, Calfornia Institute of Technology, Pasadena, CA 91125, USA
E-mail address: malabika@its.caltech.edu

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA
E-mail address: seeger@math.wisc.edu