

Math 320, Fall 2007, Homework Set 1

(due on Wednesday September 12, 2007)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
 - You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
 - Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.
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1. Let L and U be nonempty subsets of \mathbb{R} with $\mathbb{R} = L \cup U$ and such that for each $l \in L$ and $u \in U$, $l < u$. Then prove that either L has a greatest element or U has a least element.
 2. Problems from the textbook: (Chapter 1) 7, 8, 9, 16, 19.
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(For those of you who do not have the textbook yet, here are the statements of the textbook problems mentioned above).

- **(Problem 7, Chapter 1)** Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b* .)
 - (a) For any positive integer n , $b^n - 1 \geq n(b - 1)$.
 - (b) Hence $b - 1 \geq n \left(b^{\frac{1}{n}} - 1 \right)$.
 - (c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{\frac{1}{n}} < t$.
 - (d) If w is such that $b^w < y$, then $b^{w + \frac{1}{n}} < y$ for sufficiently large n ; to see this, apply part (c) with $t = yb^{-w}$.
 - (e) If $b^w > y$, then $b^{w - \frac{1}{n}} > y$ for sufficiently large n .
 - (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.
 - (g) Prove that x is unique.
- **(Problem 8, Chapter 1)** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.
- **(Problem 9, Chapter 1)** Suppose that $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order* or *lexicographic order*, for

obvious reasons.) Does this ordered set have the least upper bound property?

- **(Problem 16, Chapter 1)** Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(b) If $2r = d$, there is exactly one \mathbf{z} .

(c) If $2r < d$, there is no such \mathbf{z} .

How must these statements be modified if k is 2 or 1?

- **(Problem 19, Chapter 1)** Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$. (*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)