Math 320, Fall 2007, Homework Set 2
(due on Wednesday September 19 2007)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. We briefly discussed Cantor's construction of real numbers (via Cauchy sequences) in lecture on Wednesday September 12. Answer the following questions to fill in some of the details of this construction:

   (a) Let $\mathcal{C}$ and $\mathcal{N}$ denote the collection of all Cauchy sequences and null sequences (consisting of rationals) respectively. Show that $\mathcal{N} \subseteq \mathcal{C}$.

   (b) Verify that $\mathcal{N}$ induces an equivalence relation on $\mathcal{C}$.

   (c) Let $\mathbb{R} := \mathcal{C}/\mathcal{N}$, i.e., the collection of cosets of $\mathcal{C}$ under the equivalence relation induced by $\mathcal{N}$. Define addition and multiplication on $\mathbb{R}$ so that $\mathbb{R}$ is a field with respect to these operations.

   (d) A (nonzero) coset C in $\mathcal{C}/\mathcal{N}$ is said to be > 0 if there exists $\{a_n\} \in C$ such that $a_n > 0$ for all n. Verify that this indeed gives rise to an order relation, and that $\mathbb{R}$ equipped with this order is an ordered field.

   (e) Define a subfield $\mathbb{Q}^*$ of $\mathbb{R}$ so that $\mathbb{Q}^*$ is isomorphic to $\mathbb{Q}$. (**Definition** $\varphi : (E, <_1) \to (F, <_2)$ is an isomorphim between two ordered fields if it is a one-to-one, onto mapping that preserves the field operations and the order relation, namely,
   - (i) (one-to-one) $x \neq y$ implies $\varphi(x) \neq \varphi(y)$,
   - (ii) (onto) $\varphi(E) = F$,
   - (iii) $\varphi(x + y) = \varphi(x) + \varphi(y)$,
   - (iv) $\varphi(xy) = \varphi(x)\varphi(y)$ and
   - (v) (order-preserving) $x <_1 y$ implies $\varphi(x) <_2 \varphi(y)$

   for all x and y in E.)

   (f) (Optional) Does there exist a field isomorphism between $\mathbb{R}$ as defined above and the field of Dedekind cuts? If yes, describe the isomorphism. If not, explain why not.

2. Using the axiomatic definition of $\mathbb{R}$ (as an ordered field containing $\mathbb{Q}$ and satisfying the least upper bound property), show that every Cauchy sequence in $\mathbb{R}$ converges.

3. If $a < b$, $a, b \in \mathbb{R}$, prove that there is an irrational x such that $a < x < b$. Show that there are in fact infinitely many such irrationals.

4. Prove that $\mathbb{R}$ is uncountable.

5. Find the least upper bounds and greatest lower bounds (if they exist) of the following sets:

   (a) $S = \left\{ \frac{p}{q} : q \text{ even}, \frac{p}{q} \leq \sqrt{10} \right\}$.

   (b) $S = \left\{ \left(\frac{1}{m} + \frac{1}{n} \right)^{m+n} : m, n \in \mathbb{N} \right\}$.

   (c) $S = \left\{ \begin{array}{l} \text{all numbers that may be written as infinite decimal} \\ \text{fractions of the form } .\alpha_1 \alpha_2 \cdots ,: \alpha_i \text{ odd integer for all } i \end{array} \right\}$.