Math 320, Fall 2007, Homework Set 4
(due on Wednesday October 3 2007)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Consider $\mathbb{R}^\infty$, the vector space of all real sequences. Show that the expression
$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$
defines a metric on $\mathbb{R}^\infty$. Is this metric induced by a norm?
2. In class, we defined the normed vector space $\ell^p(\mathbb{N})$ for $1 \le p < \infty$. There is also the normed space $\ell^\infty(\mathbb{N})$, the vector space of all bounded sequences, namely sequences $\mathbf{x} = \{x_n\}$ such that $\sup_n |x_n| < \infty$. You can check (but need not include in the submitted homework) that $||\mathbf{x}||_\infty = \sup_n |x_n|$ is in fact a norm. With this definition in place, answer the following questions.

   Let F be the set of all $\mathbf{x} = \{x_n\} \in \ell^\infty(\mathbb{N})$ such that $x_n = 0$ for all but finitely many n. Is F open? closed?
3. In Homework 1, you found the least upper bound and greatest lower bound of the set
$$S = \left\{ \begin{array}{l} \alpha \in \mathbb{R} : \alpha \text{ can be written as an infinite decimal} \\ \text{expansion of the form } .\alpha_1\alpha_2\alpha_3\cdots,\ \alpha_i \text{ odd} \end{array} \right\}.$$
Let us investigate a few more properties of S. For example,
(a) Is every point of S a limit point of S?
(b) Is S closed?
4. Some of our intuition (from $\mathbb{R}$) about open and closed sets does not carry over to general normed linear spaces. For example, show that the set $A = \{\mathbf{x} \in \ell^2(\mathbb{N}) : |x_n| \le \frac{1}{n},\ n = 1, 2, \cdots\}$ is a closed set in $\ell^2(\mathbb{N})$ but that $B = \{\mathbf{x} \in \ell^2(\mathbb{N}) : |x_n| < \frac{1}{n},\ n = 1, 2, \cdots\}$ is *not* open!

5. A set that is simultaneously closed and open is sometimes called a *clopen set*. For example, we have already seen in class that every set in the discrete space $\mathbb{N}$ is clopen. Show that $\mathbb{R}$ has no nontrivial clopen sets. Find a metric space where some (but not all) sets are clopen.
6. A set A is said to be *dense* (or *everywhere dense*) in a metric space (M, d) if every point in $M \setminus A$ is a limit point of A, or equivalently $\bar{A} = M$. For example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$. A metric space is *separable* if it contains a countable dense subset. Thus $\mathbb{R}$ (or for that matter any $\mathbb{R}^n$, why?) is separable.
   (a) Show that A is dense in M if and only if A^c has empty interior.
   (b) Are $\ell^2(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ separable?
   (c) Can $\mathbb{R}$ have a proper open dense subset? If yes, find one. If not, explain why not.
7. (Optional) Let $\mathfrak{C}_0$ denote the set of all sequences in ℓ^∞ that converge to zero. Show that $\mathfrak{C}_0$ is a closed subspace of ℓ^∞.
8. (Optional) We used a result in class called *Hölder's inequality* to prove that $(\ell^p(\mathbb{N}), ||\cdot||_p)$ is a normed linear space. Follow the steps outlined below to deduce a proof of this result. In this problem, p and q are always numbers satisfying $1 < p, q < \infty$, $1/p + 1/q = 1$.
   (a) For any a, $b \geq 0$ show that $ab \leq a^p/p + b^q/q$, with equality if and only if $a^{p-1} = b$. This is known as *Young's inequality*.
   (b) Use Young's inequality to prove Hölder's inequality, namely, given $\mathbf{x} \in \ell^p$, $\mathbf{y} \in \ell^q$, show that

$$\sum_{i=1}^{\infty} |x_i y_i| \leq ||\mathbf{x}||_p ||\mathbf{y}||_q.$$