Math 320, Fall 2007, Homework Sets 5 and 6 (due on Wednesday October 17 2007)

Instructions

- **Note that there is no homework due on Wednesday October 10.**
- Homework will be collected at the end of lecture on Wednesday October 17.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Let (M, d) be a metric space, and $x \in M$. Suppose that $\{x_n\}$ is a sequence in M with the property that every subsequence in M has a further subsequence that converges to x. Show that x_n converges to x.
2. Show that every open set in $\mathbb{R}$ is the union of countably many open intervals with rational endpoints. Use this to show that the collection $\mathcal{U}$ of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself. (This indicates that "most" subsets of $\mathbb{R}$ are neither open nor closed – why?)
3. Let $f : (M, d) \to (N, \rho)$. We say f is an *open map* if f maps open sets to open sets. Similarly f is called a *closed map* if it maps closed sets to closed sets. Give examples to show that:
   (a) There are continuous maps that are not open and open maps that are not continuous.
   (b) There are continuous maps that are not closed and closed maps that are not continuous.
   (c) An open continuous map need not be closed, even if it is onto.
   (d) A closed continuous map need not be open, even if it is onto.
4. Let d be a metric on an infinite set M. Prove that there is an open set U in M such that both U and U^c are infinite.
5. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. (This is Problem 24 on page 45

of the textbook. Rudin gives a hint for this problem that you may want to look up).

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin\left(\frac{1}{q}\right) & \text{if } x = \frac{p}{q} \text{ is rational, } \gcd(p, q) = 1. \end{cases}$$

   Find all points x where f is continuous.

7. The *Hilbert cube* $\mathbb{H}^\infty$ is the collection of all real sequences $\mathbf{x} = \{x_n\}$ with $|x_n| \leq 1$ for $n \in \mathbb{N}$. You should check, but need not submit, that

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

   defines a metric on $\mathbb{H}^\infty$.

   (a) Is $\mathbb{H}^\infty$ separable?

   (b) Let (M, ρ) be a separable metric space such that $\rho(x, y) \leq 1$ for every x, $y \in M$. Given a countable dence set $\{x_n : n \geq 1\}$ in M, define a map $f : M \to \mathbb{H}^\infty$ by $f(x) = \{\rho(x, x_n) : n \in \mathbb{N}\}$. Prove that f is a homeomorphism into $\mathbb{H}^\infty$.

   (c) Is every separable metric space (M, ρ) homeomorphic to a subset of the Hilbert cube?