Math 320, Fall 2007, Homework Set 7 (due on Wednesday October 31 2007)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. We sketched a proof of the following result in class. Fill in the omitted details in that proof.
   "Let E be a subset of a metric space M. If U and V are disjoint open sets in E, then there are disjoint open sets A and B in M such that $U = A \cap E$ and $V = B \cap E$."
2. Prove that each nonempty open set U in $\mathbb{R}$ can be uniquely written as a countable union of disjoint open intervals. (Compare this result with an earlier homework problem of a similar flavor.)
3. (a) If $\mathcal{C}$ is a collection of connected subsets of M, all having a point in common, prove that $\bigcup\{A : A \in \mathcal{C}\}$ is connected. (Hint: As a first step, try with two sets). Use this to give another proof that $\mathbb{R}$ is connected.
   (b) If every pair of points in M is contained in some connected set, show that M is itself connected.
4. If M is connected and has at least two points, can M be countable?
5. If $f : [a, b] \to [a, b]$ is continuous, show that f has a fixed point; that is, show that there is some point x in $[a, b]$ with $f(x) = x$.
6. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that f takes on no value more than twice. Show that f takes on some value exactly once. [Hint: Either the maximum or the minimum value occurs exactly once.] Consequently f is piecewise monotone.
   (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ takes on each of its values exactly twice; that is, for each $y \in \mathbb{R}$, the set $\{x : y = f(x)\}$ has either 0 or 2 elements. Show that f is discontinuous at infinitely many points.
7. (Extra credit) Prove or disprove: there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(\mathbb{Q}) \subset \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}$.