Math 320, Fall 2007, Homework Sets 9 and 10 (due on Wednesday November 21 2007)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. In this problem, we will study compactness in the metric space ℓ_2.
   (a) Is the unit ball $B = \{\mathbf{x} \in \ell_2 : ||\mathbf{x}||_2 \leq 1,\ n \in \mathbb{N}\}$ compact in ℓ_2?
   (b) How about $A = \{\mathbf{x} \in \ell_2 : |x_n| \leq 1/n,\ n \in \mathbb{N}\}$?
   (c) Find a necessary and sufficient condition on the sequence $\{c_n\}$ of non-negative numbers such that the set $\{\mathbf{x} \in \ell_2 : |x_n| \leq c_n, n \in \mathbb{N}\}$ is compact in ℓ_2.
2. Show that the Hilbert cube $\mathbb{H}^\infty$ is compact. (Look up the definition of the Hilbert cube from homework set 5-6).
3. Suppose M is compact and $f : M \to N$ is continuous, one-to-one, and onto. Prove that f is a homeomorphism.
4. If $f : \mathbb{R} \to \mathbb{R}$ is both continuous and open, show that f is strictly monotone.
5. Let M be a compact metric space and suppose that $f : M \to M$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Show that f has a fixed point. [This result is very similar to, but not the same as, the Banach contraction mapping principle that you proved in the last homework assignment. Make sure you understand the distinction between the two conditions "$d(f(x), f(y)) \leq \alpha d(x, y)$ for some $0 \leq \alpha < 1$" and "$d(f(x), f(y)) < d(x, y)$".]
6. Let F and K be disjoint, nonempty subsets of a mteric space M with F closed and K compact. Show that $d(F, K) > 0$, where $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\}$. Show that this may fail if we only assume that F and K are disjoint closed sets.
7. Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto. Is compactness

necessary in this exercise? That is, is it possible for a metric space to be isometric to a proper subset of itself? Explain.

8. Define $f : \ell_2 \to \ell_1$ by $f(\mathbf{x}) = \{x_n/n : n \in \mathbb{N}\}$. Is f continuous? Uniformly continuous?
9. Give an example of a bounded continuous map $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous. Can an unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous? Explain.
10. Prove that $f : (M, d) \to (N, \rho)$ is uniformly continuous if and only if $\rho(f(x_n), f(y_n)) \to 0$ for any pair of sequences (x_n) and (y_n) in M satisfying $d(x_n, y_n) \to 0$. [Hint : For the backward implication, assume that f is not uniformly continuous and work toward a contradiction.]