Math 542, Fall 2007, Homework Set 1 (due on Friday September 28 2007)

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. The statement of Van der Corput's Lemma that we proved in class required φ' to be monotone when $k = 1$. Does the same conclusion hold without the monotonicity assumption?
2. Recall the proof of the asymptotic development of a scalar oscillatory integral in one variable. This problem is intended to help fill in the details left as exercises in that proof.
   (a) Let $\eta \in C_0^\infty(\mathbb{R})$ and ℓ a non-negative integer. Let α be a non-negative function with the property that $\alpha(x) = 1$ for $|x| \le 1$ and $\alpha(x) = 0$ for $|x| \ge 2$. Show that for $0 < \epsilon < 1$ and $N \ge 1$ satisfying $\ell - 2N < -1$,
   $$\left| \int e^{i\lambda x^2} x^\ell \eta(x) \left[1 - \alpha\left(\frac{x}{\epsilon}\right)\right] dx \right| \le C_N \lambda^{-N} \epsilon^{\ell - 2N + 1}.$$
   (b) Prove using integration by parts that
   $$\int e^{i\lambda x^2} g(x)\, dx = O(\lambda^{-N})$$
   whenever $g \in \mathcal{S}$ and g vanishes near the origin.
   (c) Show that our proof in the case $k = 2$ yields
   $$a_0 = c\,(\varphi''(x_0))^{-\frac{1}{2}}\,\psi(x_0)$$
   as the leading coefficient of the asymptotic expansion, where c is an absolute constant.
   (d) Work through the details of the proof for $k > 2$.
3. Examine the proof of the multivariate scaling principle to determine how the implicit constant $c_k(\phi)$ depends on the C^{k+1}-norm of the phase function ϕ.

4. Show that the Bessel function can also be expressed as
$$J_m(r) = \frac{(\frac{r}{2})^m}{\Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^{1} e^{irt}(1-t^2)^{m-\frac{1}{2}}\, dt.$$
5. Check that if x_0 is a nondegenerate critical point of a multivariate function φ, it is an isolated critical point.
6. Let $\chi(x)$ denote the characteristic function of the unit ball in $\mathbb{R}^n$. Show that
$$|\widehat{\chi}(\xi)| \leq C(1+|\xi|)^{-\frac{n+1}{2}}.$$
7. Check that the Fourier transform of an L^1 function is continuous, and hence defined everywhere on $\mathbb{R}^n$. Can you say the same about Fourier transforms of L^2 functions?
8. Let $T_{\alpha,\gamma}$ denote the convolution operator against the kernel
$$K_{\alpha,\gamma}(x) = e^{i|x|^{-\alpha}}|x|^{-\gamma}\phi(x),$$
where $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi \equiv 1$ for x near the origin.
   (a) Find conditions on α and γ so that $T_{\alpha,\gamma}$ is a bounded operator on $L^2(\mathbb{R}^n)$.
   (b) Discuss the behavior of the multiplier associated to $T_{\alpha,\gamma}$.