MATHEMATICS 200 April 2004 Final Exam Solutions

1) Find the distance from the point $(1, 2, 3)$ to the plane that passes through the points $(0, 1, 1)$, $(1, -1, 3)$ and $(2, 0, -1)$.

Solution. The two vectors

$$\mathbf{a} = (1, -1, 3) - (0, 1, 1) = (1, -2, 2)$$
$$\mathbf{b} = (2, 0, -1) - (0, 1, 1) = (2, -1, -2)$$

both lie inside the plane. So the vector

$$\mathbf{c} = \tfrac{1}{3}\mathbf{a} \times \mathbf{b} = \tfrac{1}{3}(6, 6, 3) = (2, 2, 1)$$

is perpendicular to the plane. The vector

$$\mathbf{d} = (1, 2, 3) - (0, 1, 1) = (1, 1, 2)$$

joins the point to the plane. So, if θ is the angle between $\mathbf{d}$ and $\mathbf{c}$, the distance is

$$|\mathbf{d}| \cos\theta = \tfrac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}|} = \tfrac{6}{\sqrt{9}} = \boxed{2}$$

2) Two sides and the enclosed angle of a triangle are measured to be $3\pm.1$m, $4\pm.1$m and $90\pm1^\circ$ respectively. The length of the third side is then computed using the cosine law $C^2 = A^2 + B^2 - 2AB\cos\theta$. What is the approximate maximum error in the computed value of C?

Solution. Let $C(A, B, \theta) = \sqrt{A^2 + B^2 - 2AB\cos\theta}$. Then $C\big(3, 4, \frac{\pi}{2}\big) = 5$. Differentiating $C^2 = A^2 + B^2 - 2AB\cos\theta$ gives

$$\begin{aligned}
2C\tfrac{\partial C}{\partial A}(A, B, \theta) &= 2A - 2B\cos\theta &\implies\quad& 10\tfrac{\partial C}{\partial A}\big(3, 4, \tfrac{\pi}{2}\big) = 6\\
2C\tfrac{\partial C}{\partial B}(A, B, \theta) &= 2B - 2A\cos\theta &\implies\quad& 10\tfrac{\partial C}{\partial B}\big(3, 4, \tfrac{\pi}{2}\big) = 8\\
2C\tfrac{\partial C}{\partial \theta}(A, B, \theta) &= 2AB\sin\theta &\implies\quad& 10\tfrac{\partial C}{\partial \theta}\big(3, 4, \tfrac{\pi}{2}\big) = 24
\end{aligned}$$

Hence the approximate maximum error in the computed value of C is

$$\begin{aligned}
|\Delta C| &\approx \big|\tfrac{\partial C}{\partial A}\big(3, 4, \tfrac{\pi}{2}\big)\Delta A + \tfrac{\partial C}{\partial B}\big(3, 4, \tfrac{\pi}{2}\big)\Delta B + \tfrac{\partial C}{\partial \theta}\big(3, 4, \tfrac{\pi}{2}\big)\Delta\theta\big|\\
&\le (0.6)(0.1) + (0.8)(0.1) + (2.4)\tfrac{\pi}{180}\\
&= \boxed{\tfrac{\pi}{75} + 0.14 \le 0.182}
\end{aligned}$$

3) A meteor strikes the ground in the heartland of Canada. Using satellite photographs, a model

$$z = f(x, y) = -\tfrac{100}{x^2+2x+4y^2+11}$$

of the resulting crater is made and a plan is drawn up to convert the site into a tourist attraction. A car park is to be built at $(4, 5)$ and a hiking trail is to be made. The trail is to start at the car park and take the steepest route to the bottom of the crater.

(a) Sketch a map of the proposed site clearly marking the car park, a few level curves for the function f and the trail.

(b) In which direction does the trail leave the car park?

Solution. (a) Since

$$z = -\tfrac{100}{x^2+2x+4y^2+11} = -\tfrac{100}{(x+1)^2+4y^2+10}$$

the bottom of the crater is at $x = -1$, $y = 0$ (where the denominator is a minimum) and the contours (level curves) are ellipses having equations $(x + 1)^2 + 4y^2 = C$. In the sketch below, the filled dot represents the bottom of the crater and the open dot represents the car park. The contours sketched are (from inside out) $z = -7.5, -5, -2.5, -1$. Note that the trail crosses the contour lines at right angles.

y

x

(b) The trail is to be parallel to

$$\nabla z = \frac{100}{(x^2+2x+4y^2+11)^2}(2x+2, 8y)$$

At the car park $\nabla z(4,5) \parallel (10,40) \parallel (1,4)$. To move towards the bottom of the crater, we should leave in the direction $\boxed{-(1,4)}$.

4) Consider the function

$$f(x,y) = 2x^3 - 6xy + y^2 + 4y$$

(a) Find and classify all of the critical points of $f(x,y)$.
(b) Find the maximum and minimum values of $f(x,y)$ in the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

Solution. (a)

$$\begin{aligned} f &= 2x^3 - 6xy + y^2 + 4y \\ f_x &= 6x^2 - 6y \qquad f_{xx} = 12x \quad f_{xy} = -6 \\ f_y &= -6x + 2y + 4 \quad f_{yy} = 2 \end{aligned}$$

The critical points are the solutions of

$$\begin{aligned} & f_x = 0 \quad f_y = 0 \\ \iff\ & y = x^2 \quad y - 3x + 2 = 0 \\ \iff\ & y = x^2 \quad x^2 - 3x + 2 = 0 \\ \iff\ & y = x^2 \quad x = 1 \text{ or } 2 \end{aligned}$$

So, there are two critical points: $(1,1)$, $(2,4)$.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1,1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2,4)$	$24 \times 2 - (-6)^2 > 0$	24	local min

y
$(1,1)$
$(0,1)$
$(1,0)$
x

(b) There are no critical points in the interior of the allowed region, so both the maximum and the minimum occur only on the boundary. The boundary consists of the line segments (i) $x = 1$, $0 \le y \le 1$, (ii) $y = 1$, $0 \le x \le 1$ and (iii) $y = 1 - x$, $0 \le x \le 1$.

First, we look at the part of the boundary with $x = 1$. There $f = y^2 - 2y + 2$. As $\frac{d}{dy}(y^2 - 2y + 2) = 2y - 2$ vanishes only at $y = 1$, the max and min of $y^2 - 2y + 2$ for $0 \le y \le 1$ must occur either at $y = 0$, where $f = 2$, or at $y = 1$, where $f = 1$.

Next, we look at the part of the boundary with $y = 1$. There $f = 2x^3 - 6x + 5$. As $\frac{d}{dx}(2x^3 - 6x + 5) = 6x^2 - 6$, the max and min of $2x^3 - 6x + 5$ for $0 \le x \le 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 1$.

Next, we look at the part of the boundary with $y = 1-x$. There $f = 2x^3 - 6x(1-x) + (1-x)^2 + 4(1-x) = 2x^3 + 7x^2 - 12x + 5$. As $\frac{d}{dx}(2x^3 + 7x^2 - 12x + 5) = 6x^2 + 14x - 12 = 2(3x-2)(x+3)$, the max and min of $2x^3 + 7x^2 - 12x + 5$ for $0 \le x \le 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 2$, or at $x = \frac{2}{3}$, where $f = 2(\frac{8}{27}) - 6(\frac{2}{3})(\frac{1}{3}) + \frac{1}{9} + \frac{4}{3} = \frac{16-36+3+36}{27} = \frac{19}{27}$.

All together, we have the following candidates for max and min

point	$(1,0)$	$(1,1)$	$(0,1)$	$(\frac{2}{3},\frac{1}{3})$
value of f	2	1	5	$\frac{19}{27}$

The largest and smallest values of f in this table are $\boxed{\text{min}= \frac{19}{27},\ \text{max}= 5}$.

5 Let S be the surface

$$xy - 2x + yz + x^2 + y^2 + z^3 = 7$$

(a) Find the tangent plane and normal line to the surface S at the point $(0,2,1)$.

(b) The equation defining S implicitly defines z as a function of x and y for (x,y,z) near $(0,2,1)$. Find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Evaluate $\frac{\partial z}{\partial y}$ at $(x,y,z) = (0,2,1)$.

(c) Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$.

Solution. (a) A normal vector to the surface at $(0,2,1)$ is

$$\begin{aligned}\boldsymbol{\nabla}\big(xy - 2x + yz + x^2 + y^2 + z^3 - 7\big)\big|_{(0,2,1)} &= \big(y - 2 + 2x, x + z + 2y, y + 3z^2\big)\big|_{(0,2,1)} \\ &= (0,5,5)\end{aligned}$$

So the tangent plane is

$$0(x-0) + 5(y-2) + 5(z-1) = 0 \text{ or } \boxed{y + z = 3}$$

The vector parametric equations for the normal line are

$$\boxed{\mathbf{r}(t) = (0,2,1) + t(0,5,5)}$$

(b) Differentiating

$$xy - 2x + yz(x,y) + x^2 + y^2 + z(x,y)^3 = 7$$

gives

$$\begin{aligned} y - 2 + yz_x(x,y) + 2x + 3z(x,y)^2 z_x(x,y) = 0 \quad &\Longrightarrow \quad z_x(x,y) = \tfrac{2-2x-y}{y+3z(x,y)^2} \\ x + z(x,y) + yz_y(x,y) + 2y + 3z(x,y)^2 z_y(x,y) = 0 \quad &\Longrightarrow \quad z_y(x,y) = -\tfrac{x+2y+z(x,y)}{y+3z(x,y)^2}\end{aligned}$$

In particular, at $(0,2,1)$, $\boxed{z_y(0,2) = -1}$.

(c) Differentiating z_x with respect to y gives

$$\begin{aligned} z_{xy}(x,y) &= -\tfrac{1}{y+3z(x,y)^2} - \tfrac{2-2x-y}{[y+3z(x,y)^2]^2}\big(1 + 6z(x,y)z_y(x,y)\big) \\ &= \boxed{-\tfrac{1}{y+3z(x,y)^2} - \tfrac{2-2x-y}{[y+3z(x,y)^2]^2}\big(1 - 6z(x,y)\tfrac{x+2y+z(x,y)}{y+3z(x,y)^2}\big)}\end{aligned}$$

As an alternate solution, we could also differentiate z_y with respect to x. This gives

$$\begin{aligned} z_{yx}(x,y) &= -\tfrac{1+z_x(x,y)}{y+3z(x,y)^2} + \tfrac{x+2y+z(x,y)}{[y+3z(x,y)^2]^2} 6z(x,y)z_x(x,y) \\ &= \boxed{-\tfrac{1}{y+3z(x,y)^2}\big(1 + \tfrac{2-2x-y}{y+3z(x,y)^2}\big) + \tfrac{x+2y+z(x,y)}{[y+3z(x,y)^2]^2} 6z(x,y)\tfrac{2-2x-y}{y+3z(x,y)^2}}\end{aligned}$$

6 Find the points on the ellipse $2x^2 + 4xy + 5y^2 = 30$ which are closest to and farthest from the origin.

Solution. Let (x, y) be a point on $2x^2 + 4xy + 5y^2 = 30$. We wish to maximize and minimize $x^2 + y^2$ subject to $2x^2 + 4xy + 5y^2 = 30$. Define $L(x, y, \lambda) = x^2 + y^2 - \lambda(2x^2 + 4xy + 5y^2 - 30)$. Then

$$\begin{aligned}
0 = L_x &= 2x - \lambda(4x + 4y) && \implies (1 - 2\lambda)x - 2\lambda y \quad = 0 && (1)\\
0 = L_y &= 2y - \lambda(4x + 10y) && \implies -2\lambda x + (1 - 5\lambda)y = 0 && (2)\\
0 = L_\lambda &= 2x^2 + 4xy + 5y^2 - 30
\end{aligned}$$

Note that λ cannot be zero because if it is, (1) forces $x = 0$ and (2) forces $y = 0$, but $(0, 0)$ is not on the ellipse. So equation (1) gives $y = \frac{1-2\lambda}{2\lambda}x$. Subbing this into equation (2) gives $-2\lambda x + \frac{(1-5\lambda)(1-2\lambda)}{2\lambda}x = 0$. To get a nonzero (x, y) we need

$$-2\lambda + \tfrac{(1-5\lambda)(1-2\lambda)}{2\lambda} = 0 \iff 0 = -4\lambda^2 + (1 - 5\lambda)(1 - 2\lambda) = 6\lambda^2 - 7\lambda + 1 = (6\lambda - 1)(\lambda - 1)$$

So λ must be either 1 or $\frac{1}{6}$. Subbing these into either (1) or (2) gives

$$\begin{aligned}
\lambda = 1 &\implies -x - 2y = 0 \implies x = -2y \implies 8y^2 - 8y^2 + 5y^2 \quad = 30 \implies y = \pm\sqrt{6}\\
\lambda = \tfrac{1}{6} &\implies \tfrac{2}{3}x - \tfrac{1}{3}y = 0 \implies y = 2x \quad \implies 2x^2 + 8x^2 + 20x^2 = 30 \implies x = \pm 1
\end{aligned}$$

The farthest points are $\boxed{\pm\sqrt{6}(-2, 1)}$. The nearest points are $\boxed{\pm(1, 2)}$.

7) Consider the integral

$$\int_0^1 \int_x^1 e^{x/y}\ dy\ dx$$

(a) Sketch the domain of integration.
(b) Evaluate the integral by reversing the order of integration.

Solution. (a) The domain of integration is sketched in

y $\quad$ $x = y$

1 $\quad$ $y = 1$

1 x

(b) Reversing the order of integration gives

$$\int_0^1 dy \int_0^y dx\ e^{x/y} = \int_0^1 dy\ \left[ye^{x/y}\right]_0^y = \int_0^1 dy\ y(e - 1) = \boxed{\tfrac{1}{2}(e - 1)}$$

8) A solid is bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. It has density $\rho(x, y, z) = x^2 + y^2$.

(a) Express the mass m of the solid as a triple integral in cylindrical coordinates.

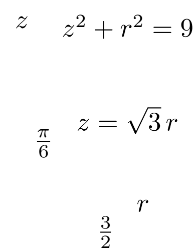
(b) Express the mass m of the solid as a triple integral in spherical coordinates.

(c) Evaluate m.

Solution. (a) In cylindrical coordinates, the density of is $\rho = x^2 + y^2 = r^2$, the bottom of the solid is at $z = \sqrt{3x^2 + 3y^2} = \sqrt{3}\,r$ and the top of the solid is at $z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$. The top and bottom meet when $\sqrt{3}\,r = \sqrt{9 - r^2} \iff 3r^2 = 9 - r^2 \iff 4r^2 = 9 \iff r = \frac{3}{2}$. The mass is

$$m = \int_0^{2\pi} d\theta \int_0^{3/2} dr\ r \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} dz\ r^2$$

(b) In spherical coordinates, the density of is $\rho = x^2 + y^2 = R^2 \sin^2 \varphi$, the bottom of the solid is at $z = \sqrt{3}\,r \iff R\cos\varphi = \sqrt{3}\,R\sin\varphi \iff \tan\varphi = \frac{1}{\sqrt{3}} \iff \varphi = \frac{\pi}{6}$ and the top of the solid is at $x^2 + y^2 + z^2 = R^2 = 9$. The mass is



$$m = \int_0^{2\pi} d\theta \int_0^{\pi/6} d\varphi \int_0^3 dR\ \big(R^2 \sin\varphi\big)\big(R^2 \sin^2\varphi\big)$$

(c) Making the change of variables $s = \cos\varphi$, $ds = -\sin\varphi\ d\varphi$,

$$\begin{aligned}
m &= \int_0^{2\pi} d\theta \int_0^{\pi/6} d\varphi \int_0^3 dR\ R^4 \sin\varphi\big(1 - \cos^2\varphi\big) \\
&= \tfrac{3^5}{5} \int_0^{2\pi} d\theta \int_0^{\pi/6} d\varphi\ \sin\varphi\big(1 - \cos^2\varphi\big) \\
&= -\tfrac{3^5}{5} \int_0^{2\pi} d\theta \int_1^{\sqrt{3}/2} ds\ (1 - s^2) \\
&= -\tfrac{3^5}{5} \int_0^{2\pi} d\theta\ \big[s - \tfrac{s^3}{3}\big]_1^{\sqrt{3}/2} \\
&= 2\pi \tfrac{3^5}{5}\big[1 - \tfrac{1}{3} - \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{8}\big] \\
&= \boxed{2\pi \tfrac{3^5}{5}\big[\tfrac{2}{3} - \tfrac{3\sqrt{3}}{8}\big]}
\end{aligned}$$

As an alternate solution, we can also evaluate the integral of part (a).

$$\begin{aligned}
m &= \int_0^{2\pi} d\theta \int_0^{3/2} dr\ r \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} dz\ r^2 \\
&= \int_0^{2\pi} d\theta \int_0^{3/2} dr\ r^3\big(\sqrt{9 - r^2} - \sqrt{3}\,r\big) \\
&= 2\pi \int_0^{3/2} dr\ r^3\big(\sqrt{9 - r^2} - \sqrt{3}\,r\big)
\end{aligned}$$

The second term

$$-2\pi \int_0^{3/2} dr\ \sqrt{3}\,r^4 = -2\pi\sqrt{3}\ \tfrac{r^5}{5}\Big|_0^{3/2} = -2\pi\sqrt{3}\ \tfrac{3^5}{5 \times 2^5}$$

For the first term, we substitute $s = 9 - r^2$, $ds = -2r\,dr$.

$$\begin{aligned}
2\pi \int_0^{3/2} dr\ r^3 \sqrt{9 - r^2} &= 2\pi \int_9^{27/4} \tfrac{ds}{-2}(9 - s)\sqrt{s} = -\pi\big[6s^{3/2} - \tfrac{2}{5}s^{5/2}\big]_9^{27/4} \\
&= -\pi\big[\tfrac{3^5\sqrt{3}}{4} - 2 \times 3^4 - \tfrac{3^7}{2^4 5}\sqrt{3} + 2\tfrac{3^5}{5}\big]
\end{aligned}$$

Adding

$$m = 2\pi \tfrac{3^5}{5}\big[\big(\tfrac{5}{3} - 1\big) - \sqrt{3}\big(\tfrac{1}{32} + \tfrac{5}{8} - \tfrac{9}{32}\big)\big] = \boxed{2\pi \tfrac{3^5}{5}\big[\tfrac{2}{3} - \tfrac{3\sqrt{3}}{8}\big]}$$