Math 263 Assignment 3 Solutions

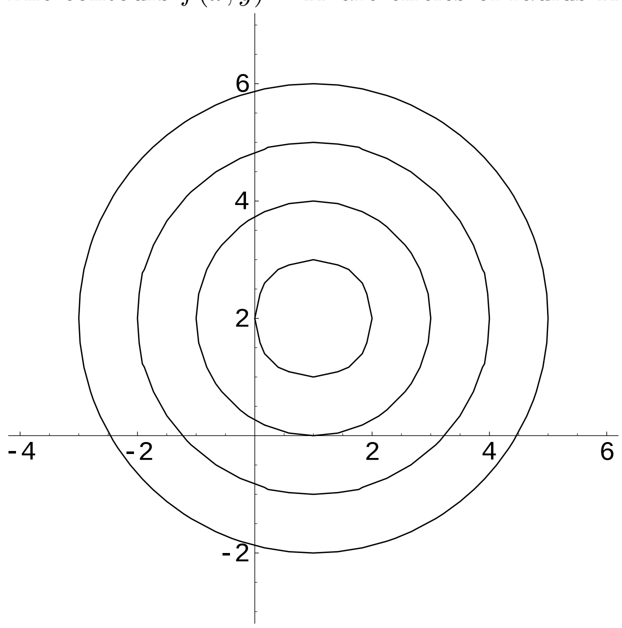
1. (a) Draw a contour diagram for the function $f(x,y) = \sqrt{(x-1)^2 + (y-2)^2}$. Indicate the contours $f(x,y) = 1, 2, 3$ and 4.

   (b) Calculate $\nabla f(2,3)$ and indicate this vector on your diagram.

   (c) Consider $z = f(x,y)$. Find the equation of the tangent plane to $f(x,y)$ at the point $(2,3)$.

Solution:

(a) The contours $f(x,y) = K$ are circles of radius K centred at $(1,2)$.



(b)

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}} \\ \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} \end{pmatrix}.$$

Therefore $\nabla f(2,3) = (1/\sqrt{2}, 1/\sqrt{2})$.

(c) The tangent plane is given by

$$z = f(2,3) + \nabla f(2,3).\begin{pmatrix} x-2 \\ y-3 \end{pmatrix} = \sqrt{2} + \frac{1}{\sqrt{2}}(x-2) + \frac{1}{\sqrt{2}}(y-3).$$

In standard equation form, this is

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - z = -\sqrt{2} + \frac{5}{\sqrt{2}}$$

.

2. A function $z = f(x, y)$ is called *harmonic* if it satisfies this equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

This is called *Laplace's Equation.* Determine whether or not the following functions are harmonic:

(a) $z = \sqrt{x^2 + y^2}$

(b) $e^{-x} \sin y$

(c) $3x^2y - y^3$

Solution:

(a) Simply differentiate twice with respect to x to get $\partial^2 z/\partial x^2$ and twice with respect to y to get $\partial^2 z/\partial y^2$. You will get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{\sqrt{x^2 + y^2}} \neq 0$$

so this function is not harmonic.

(b)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x} \sin y - e^{-x} \sin y = 0$$

so this function is harmonic.

(c)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 6y - 6y = 0$$

so this function is harmonic.

3. In each case, give an example of an appropriate function or show that no such fuction exists.

(a) A function $f(x, y)$ with continuous second order partial derivatives and which satisfies $\frac{\partial f}{\partial x} = 6xy^2$ and $\frac{\partial f}{\partial y} = 8x^2y$.

(b) A function $g(x, y)$ satisfying the equations $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 2xy$.

Solution:

(a) We know that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Taking $\frac{\partial f}{\partial x} = 6xy^2$ and differentiating with respect to y we get

$$\frac{\partial^2 f}{\partial y \partial x} = 12xy.$$

Similarly, differentiating $\frac{\partial f}{\partial y} = 8x^2y$ with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = 16xy.$$

Since these mixed partial derivatives are not equal, we conclude that there is no such function $f(x, y)$.

(b) If we repeat the argument from part (a), we get $\frac{\partial^2 f}{\partial y \partial x} = 2x \neq \frac{\partial^2 f}{\partial x \partial y} = 2y$, so again we find there is no such function $f(x, y)$.

4. Use the appropriate version of the chain rule to compute the following:

   (a) dw/dt at $t = 3$, where $w = \ln(x^2 + y^2 + z^2), x = \cos t, y = \sin t$, and $z = 4\sqrt{t}$.

   (b) $\partial z/\partial u$ and $\partial z/\partial v$, where $z = xy, x = u \cos v$, and $y = u \sin v$.

Solution:

(a)

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \frac{-2x \sin t}{x^2 + y^2 + z^2} + \frac{2y \cos t}{x^2 + y^2 + z^2} + \frac{4zt^{-1/2}}{x^2 + y^2 + z^2} \\
&= \frac{-2 \cos t \sin t + 2 \sin t \cos t + 16}{\cos^2 t + \sin^2 t + 16t} \\
&= \frac{16}{1 + 16t}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = y \cos v + x \sin v = 2u \sin v \cos v = u \sin 2v \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -yu \sin v + xu \cos v = u^2(\cos^2 v - \sin^2 v) = u^2 \cos 2v
\end{aligned}$$

5. Suppose a duck is swimming around in a circle, with position given by $x = \cos t$ and $y = \sin t$. Suppose that the water temperature is given by $T = x^2e^y - xy^3$. Find the rate of change in temperature that the duck experiences as it passes through the point $(1/\sqrt{2}, -1/\sqrt{2})$.

Solution: We need to compute dT/dt using the chain rule:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)(\cos t)$$

To evaluate this function at the point $x = 1/\sqrt{2}, y = -1/\sqrt{2}$, we also need to know what t is. Solving for $1/\sqrt{2} = \cos t$ and $-1/\sqrt{2} = \sin t$, we find that $t = 7\pi/4$. Plugging in x, y and t into dT/dt, we find that

$$\frac{dT}{dt} = -\frac{1}{2} + \left(\frac{1}{2\sqrt{2}} + 1\right) e^{-1/\sqrt{2}}.$$

6. Compute the following using implicit differentiation:

   (a) $\partial y/\partial z$ if $e^{yz} - x^2 z \ln y = \pi$.

   (b) dy/dx if $F(x, y, x^2 - y^2) = 0$.

   Solution:

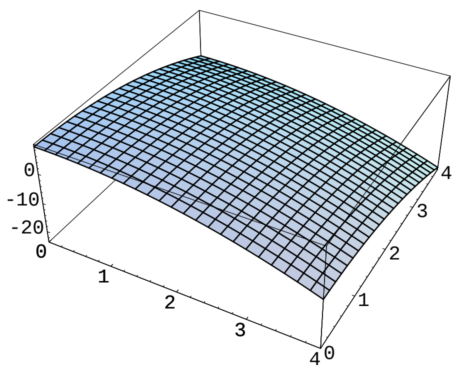
   (a) Partially differentiate both sides of the equation with respect to z. You have to have $y = y(x, z)$ and x is fixed:

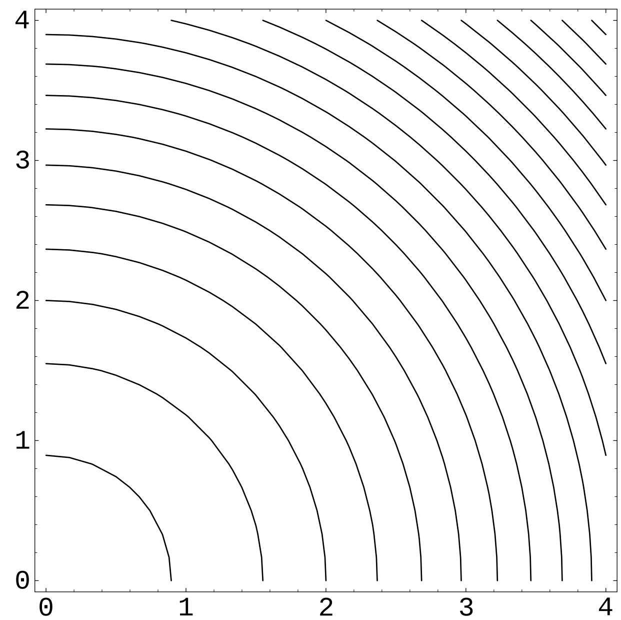
$$\begin{aligned} \frac{\partial}{\partial z}\left(e^{yz} - x^2 z \ln y\right) &= \frac{\partial}{\partial z}\pi \\ \left(z\frac{\partial y}{\partial z} + y\right) e^{yz} - \left(x^2 \ln y + x^2 z \frac{1}{y}\frac{\partial y}{\partial z}\right) &= 0 \\ \frac{\partial y}{\partial z} &= \frac{x^2 \ln y - y e^{yz}}{z e^{yz} - \frac{x^2 z}{y}}. \end{aligned}$$

   (b) We are looking for dy/dx so we have to assume that $y = y(x)$. Therefore, differentiating $F(x, y, z) = 0$ with respect to x, when $z = x^2 - y^2$, we find

$$\begin{aligned} \frac{\partial}{\partial x}F(x, y, x^2 - y^2) &= \frac{\partial}{\partial x}0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dz} + \frac{\partial F}{\partial z}\left(2x - 2y\frac{dy}{dx}\right) &= 0 \\ \frac{dy}{dx} &= \frac{-\frac{\partial F}{\partial x} - 2x\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y} - 2y\frac{\partial F}{\partial z}} \end{aligned}$$

7. The surface plot $z = f(x, y)$ and the contour diagram are shown:





Look at the point $(2, 2)$. At this point, find the sign (positive or negative) of each of the following quantities:

- $\partial f/\partial x$
- $\partial f/\partial y$
- $\partial^2 f/\partial x^2$
- $\partial^2 f/\partial y^2$
- $\partial^2 f/\partial x\partial y$

Solution: The function is decreasing as you go upwards and to the right. Therefore, $\partial f/\partial x$ and $\partial f/\partial y$ are both negative at (2,2). As you go through (2,2) in the x-direction, the contours are getting closer together. That means that $\partial f/\partial x$ is getting more negative. Therefore, $\partial^2 f/\partial x^2$ is negative. Similarly, as you go through (2,2) in the y-direction, the contours are getting closer together. That means that $\partial f/\partial y$ is also getting more negative. Therefore, $\partial^2 f/\partial y^2$ is negative too.

The most difficult one, as usual, is $\partial^2 f/\partial x\partial y$. Let's consider $\partial f/\partial y$ and see how that changes as we move through (2,2) in the x-direction. A bit to the left of (2,2), $\partial f/\partial y$ is negative. A bit to the right of (2,2), $\partial f/\partial y$ is more negative. We know this because the distance between the contours is getting smaller as we move to the right. Therefore, $\partial f/\partial y$ is decreasing as we move through (2,2) in the x-direction. This means that $\partial^2 f/\partial x\partial y$ is negative. The same conclusion can be reached by considering the change in $\partial f/\partial x$ as you move through (2,2) in the y-firection (try it!).

8. Find the equation of the tangent plane to $z = \sqrt{xy}$ at the point $(1, 1, 1)$.

Solution: $\partial z/\partial x = (y/2)(xy)^{-1/2}$ and $\partial z/\partial y = (x/2)(xy)^{-1/2}$. Therefore,

$$\begin{aligned} z &= 1 + \frac{1}{2}1^{-1/2}(x-1) + \frac{1}{2}1^{-1/2}(y-1) \\ z &= 1 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) \\ x + y - 2z &= 0 \end{aligned}$$

9. You have three resistors labeled 10Ω, 20Ω and 30Ω. Each of the resistances is guaranteed accurate to within 1%.

   (a) You connect the resistors in series, hoping to get a resistance of 60Ω *(there was a mistake in the original question here, sorry)*. Use differentials to estimate the maximum error in the resistance.

   (b) You connect the resistors in parallel, hoping to get a resistance of $\frac{60}{11}\Omega$. Use differentials to estimate the maximum error in the resistance.

Solution:

(a) Let $R_s(r_1, r_2, r_3) = r_1 + r_2 + r_3$ be the overall resistance of the series. Then the differentials dR, dr_1, dr_2 and dr_3 are connected by the formula

$$\begin{aligned} dR_s &= \frac{\partial R_s}{\partial r_1}dr_1 + \frac{\partial R_s}{\partial r_2}dr_2 + \frac{\partial R_s}{\partial r_3} \\ &= dr_1 + dr_2 + dr_3. \end{aligned}$$

$r_1 = 10\Omega, r_2 = 20\Omega, r_3 = 30\Omega, dr_1 = 0.1\Omega, dr_2 = 0.2\Omega$ and $dr_3 = 0.3\Omega$. Therefore $dR_s = 0.6\Omega$ and this is exactly equal to the maximum error in the overall resistance (if you work out the numbers, $57.4\Omega < R_s < 60.6\Omega$). Note that this illustrates that if the function is **linear**, the differential is **exact**.

(b) The resistance in parallel is given by

$$R_p(r_1, r_2, r_3) = \frac{1}{(1/r_1) + (1/r_2) + (1/r_3)}.$$

The formula for the differentials is

$$\begin{aligned} dR_p &= \frac{\partial R_p}{\partial r_1}dr_1 + \frac{\partial R_p}{\partial r_2}dr_2 + \frac{\partial R_p}{\partial r_3} \\ &= \frac{r_2^2 r_3^2}{(r_1r_2 + r_2r_3 + r_1r_3)^2}dr_1 + \frac{r_1^2 r_3^2}{(r_1r_2 + r_2r_3 + r_1r_3)^2}dr_2 + \frac{r_1^2 r_2^2}{(r_1r_2 + r_2r_3 + r_1r_3)^2}dr_3. \end{aligned}$$

Plugging in the numbers, $dR_p = 0.0545\Omega$ and this is a good estimate to the maximum error in the overall resistance (if you work out the numbers, $5.4\Omega < R_p < 5.509\Omega$).