Math 263 Assignment 4 Solutions

1) If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find the quantities $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$ and $\frac{\partial^2 z}{\partial r \partial \theta}$.

Solution. By the chain rule,

$$\begin{aligned}
\frac{\partial z}{\partial r} &= f_x(r\cos\theta, r\sin\theta)\cos\theta + f_y(r\cos\theta, r\sin\theta)\sin\theta, \\
\frac{\partial z}{\partial \theta} &= -f_x(r\cos\theta, r\sin\theta)r\sin\theta + f_y(r\cos\theta, r\sin\theta)r\cos\theta, \\
\frac{\partial^2 z}{\partial r \partial \theta} &= -f_x\sin\theta + (-f_{xx}r\sin\theta + f_{xy}r\cos\theta)\cos\theta + f_y\cos\theta \\
&\qquad + (-f_{yx}r\sin\theta + f_{yy}r\cos\theta)\sin\theta \\
&= \frac{1}{r}\frac{\partial z}{\partial \theta} + r\cos\theta\sin\theta(f_{yy} - f_{xx}) + f_{xy}r\cos 2\theta.
\end{aligned}$$

□

2) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

Solution. Let $(x(t), y(t), z(t))$ be the parametric representation of the ellipse. Then the equation of the tangent line at the point $(x(t_0), y(t_0), z(t_0)) = (1, 2, 1)$ is given by

$$x - 1 = at, \quad y - 2 = bt, \quad z - 1 = ct,$$

where $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a vector parallel to the tangent vector $x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$. We therefore need to find a, b and c, or equivalently $x'(t_0), y'(t_0), z'(t_0)$.

Now any point $(x(t), y(t), z(t))$ on the ellipse must satisfy:

$$y(t) + z(t) = 3, \qquad x(t)^2 + y(t)^2 = 5.$$

Implicitly differentiating the equations above with respect to t, we obtain

$$y'(t) + z'(t) = 0, 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

Plugging in $t = t_0$ into the equations and solving we obtain

$$y'(t_0) + z'(t_0) = 0, \quad 2x'(t_0) + 4y'(t_0) = 0, \quad \text{or} \quad x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k} = (-2\mathbf{i} + \mathbf{j} - \mathbf{k})y'(t_0).$$

We may therefore choose $a = -2, b = 1, c = -1$, obtaining the equation for the tangent as

$$x = 1 - 2t, \quad y = 2 + t, \quad z = 1 - t.$$

□

3) Find the absolute maximum and minimum values of

$$f(x,y) = x^3 - 3x - y^3 + 12y$$

over the quadrilateral with vertices $(-2,3), (2,3), (2,2), (-2,-2)$.

Solution. $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$, and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$.

Let L_1, L_4, L_3 and L_2 denote the line segments from $(-2,3) \to (-2,-2) \to (2,2) \to (2,3) \to (-2,3)$ in that order. We need to find the maximum and minimum values of f along each of these segments.

- Along L_1: $x = -2$ and $f(2,y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$.
- Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(-1,3) = f(2,3) = 11$.
- Along L_3 $y = 3$ and $f(x,3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = -1$ and $x = -2$ where $f(1,3) = f(-2,3) = 7$.
- Along L_4: $y = x$ and $f(x,x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2,2) = 18$ and a minimum at $x = -2$ where $f(-2,-2) = -18$.

In summary, the absolute maximum value of f on D is $f(2,2) = 18$ and the minimum is $f(-2,-2) = -18$. □

4) A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of 10 units/m^2 per day, the north and south walls at a rate of 8 units/m^2 per day, the floor at a rate of 1 unit/m^2 per day and the roof at the rate of 5 units/m^2 per day. Each wall must be at least 30 meters long, the height must be at least 4 m, and the volume must be exactly 4000 m^3.

(a) Find and sketch the domain of heat loss as a function of the lengths of the sides.

(b) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)

(c) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

Solution. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by

$$h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz.$$

The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = 4000/(xy)$ to obtain the heat loss function

$$h(x,y) = 6xy + \frac{80,000}{x} + \frac{64,000}{y}.$$

(a) Since $4000/(xy) \geq 4$, $xy \leq 1000$, i.e., $y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is

$$D = \{(x,y) : x \geq 30, 30 \leq y \leq \frac{1000}{x}\}.$$

This is the region bounded from below by the horizontal line segment from $(30,30)$ to $(\frac{100}{3},30)$ (let us call this line L_1), from the right by the portion of the hyperbola $y = \frac{1000}{x}$ from $(30,\frac{100}{3})$ to $(\frac{100}{3},30)$ (we call this curve L_2) and from the left by the vertical line segment from $(30,30)$ to $(30,\frac{100}{3})$ (denote this by L_3).

(b) $H_x = 6y - 80,000x^{-2}$, $h_y = 6x - 64,000y^{-2}$. $h_x = 0$ implies $6x^2y = 80,000$, or, $y = 80,000/(6x^2)$. Substituting this in to $h_y = 0$ gives

$$6x = 64,000\left(\frac{6x^2}{80,000}\right)^2, \quad \text{so } x = 10\left(\frac{50}{3}\right)^{\frac{1}{3}} \approx 25.54, y = \frac{80}{60^{\frac{1}{3}}} \approx 20.43.$$

We observe that this critical point is not in D. Nest we check the boundary of D.

- On L_1: $y = 30$, $h(x,30 = 180x + \frac{80,000}{x} + \frac{6400}{3}$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x,30) = 180 - 80,000/x^2 > 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x,30)$ is an increasing function with minimum $h(30,30) = 10,200$ and maximum $h(\frac{100}{3},30) \approx 10,533$.
- On L_2: $y = \frac{1000}{x}$, $h(x,\frac{1000}{x}) = 6000 + 64x + \frac{80,000}{x}$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x,\frac{1000}{x}) = 64 - 80,000/x^2 < 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x,1000/x)$ is a decreasing function with minimum $h(\frac{100}{3},30) \approx 10,533$ and maximum $h(30,\frac{100}{3}) \approx 10,587$.
- On L_3: $x = 30$, $h(30,y) = 180y + 64,000/y + 8000/3$, $30 \leq y \leq \frac{100}{3}$. $h'(30,y) = 180 - 64,000/y^2 > 0$ for $30 \leq y \leq \frac{100}{3}$, so $h(30,y)$ is an increasing function of y with minimum $h(30,30) = 10,200$ and maximum $h(30,\frac{100}{3}) \approx 10,587$.

Thus the absolute minimum of h is $h(30,30) = 10,200$ and the dimensions of the building that minimize heat loss are walls 30 m in length and height $4000/30^2 = 40/9 \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local and absolute minimum is approximately $h(25.54,20.43) \approx 9396$. So a building of volume 4000 m^3 with dimensions $x \approx 25.54$ m, $y = 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

□

5) A hiker stands on a hill whose shape is given by $z = x^2 - 2x + y^2 - 4y$, where the positive x-axis points east and the positive y-axis points north. He measures that he would climb the steepest path if he proceeds northeast. Find the coordinates of the point where the hiker is standing.

Solution. Setting $f(x,y) = x^2 - 2x + y^2 - 4y$ we observe that

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = (2x-2)\mathbf{i} + (2y-4)\mathbf{j},$$

which is the direction of fastest change. This vector will point in the north-east direction $\mathbf{i}+\mathbf{j}$ if $2x-2 = 2y-4$, or $y = x+1$. Therefore the hiker could be standing anywhere on a curve on the hill, given by the parametric equations

$$y = x+1, \quad z = 2(x-1)^2 - 5.$$

□

6) Consider the surface $xyz = 1$. Choose a point $P = (x_0, y_0, z_0)$ in the first octant (so $x > 0, y > 0, z > 0$) and take the tangent plane to the surface at that point. Now consider the pyramid-shaped volume that is bounded by $x = 0, y = 0, z = 0$ and that tangent plane. Show that the volume of the pyramid is the same no matter which point P you choose.

Solution. Let a, b, c denote the x, y and z intercepts of the tangent plane to the surface respectively. Then the base of the pyramid is a right triangle with vertices at the origin, $(a, 0, 0)$ and $(0, b, 0)$. The height of the pyramid is c. The volume of the pyramid is therefore $\frac{1}{3}(\frac{1}{2}ab)c = \frac{1}{6}abc$. In other words, we need to check that the quantity abc is independent of (x_0, y_0, z_0).

Let $F(x,y,z) = xyz - 1$. Then $F_x = yz, F_y = zx, F_z = xy$, giving rise to the following equation of the tangent plane

$$y_0z_0(x - x_0) + z_0x_0(y - y_0) + x_0y_0(z - z_0) = 0.$$

This implies that $a = (y_0z_0)^{-1}$, $b = (z_0x_0)^{-1}$, $c = (x_0y_0)^{-1}$; or

$$abc = \left(\frac{1}{x_0y_0z_0}\right)^2 = 1, \text{ which is a constant, as claimed.}$$

□

7) If a sound with frequency f_s is produced by a source traveling along a line with speed v_s and an observer is traveling with speed v_0 along the same line in the opposite direction toward the source, then the Doppler effect dictates that the frequency of the sound heard by the observer is

$$f_0 = \left(\frac{c + v_0}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. Suppose that at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

Solution. $f_0 = f_s(c+v_0)/(c-v_s) = 460(332+34)/(332-40) \approx 576.6$ Hz. By the chain rule,

$$\begin{aligned} \frac{df_0}{dt} &= \frac{\partial f_0}{\partial v_0}\frac{dv_0}{dt} + \frac{\partial f_0}{\partial v_s}\frac{dv_s}{dt} = \frac{f_s}{(c-v_s)}\frac{dv_0}{dt} + \frac{(c+v_0)f_s}{(c-v_s)^2}\frac{dv_s}{dt} \\ &= \frac{460}{332-40}(1.2) + \frac{(332+34)(460)}{(332-40)^2}(1.4) = 4.65\text{Hz/s}. \end{aligned}$$

□