Math 263 Assignment 5 SOLUTIONS

1. The temperature at all points in the disc $x^2 + y^2 \le 1$ is $T(x, y) = (x + y)e^{-x^2-y^2}$. Find the maximum and minimum temperatures on the disc.

   SOLUTION:

   $T(x, y) = (x+y)e^{-x^2-y^2}$ $\quad T_x(x, y) = (1-2x^2-2xy)e^{-x^2-y^2}$ $\quad T_y(x, y) = (1-2xy-2y^2)e^{-x^2-y^2}$

   First, we find the critical points

   $$\begin{aligned} T_x = 0 &\iff 2x(x+y) = 1 \\ T_y = 0 &\iff 2y(x+y) = 1 \end{aligned}$$

   As $x+y$ may not vanish, this forces $x = y$ and then $x = y = \pm 1/2$. So the only critical points are $(1/2, 1/2)$ and $(-1/2, -1/2)$.

   On the boundary $x = \cos t$ and $y = \sin t$, so $T = (\cos t + \sin t)e^{-1}$. This is a periodic function and so takes its max and min at zeroes of $\frac{dT}{dt} = \big(-\sin t + \cos t\big)e^{-1}$. That is, when $\sin t = \cos t$, which forces $\sin t = \cos t = \pm\frac{1}{\sqrt{2}}$. All together, we have the following candidates for max and min

point	$(1/2, 1/2)$	$(-1/2, -1/2)$	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
value of f	$\frac{1}{\sqrt{e}} \approx 0.61$	$-\frac{1}{\sqrt{e}}$	$\frac{\sqrt{2}}{e} \approx 0.52$	$-\frac{\sqrt{2}}{e}$

   The largest and smallest values of T in this table are min$= -\frac{1}{\sqrt{e}}$, max$= \frac{1}{\sqrt{e}}$.

2. Find the high and low points of the surface $z = \sqrt{x^2 + y^2}$ with (x, y) varying over the square $|x| \le 1$, $|y| \le 1$. Discuss the values of z_x, z_y there. Do not evaluate any derivatives in answering this question.

   SOLUTION: The surface is a cone. The minimum height is at $(0, 0, 0)$. The cone has a point there and the derivatives z_x and z_y do not exist. The maximum height is achieved when (x, y) is as far as possible from $(0, 0)$. The highest points are at $(\pm 1, \pm 1, \sqrt{2})$. There z_x and z_y exist but are not zero. These points would not be the highest points if it were not for the restriction $|x|, |y| \le 1$.

3. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 1$.

   SOLUTION: Define $L(x, y, z, \lambda) = x + y - z - \lambda(x^2 + y^2 + z^2 - 1)$. Then

   $$\begin{aligned} 0 = L_x &= 1 - 2\lambda x &&\implies x = \frac{1}{2\lambda} \\ 0 = L_y &= 1 - 2\lambda y &&\implies y = \frac{1}{2\lambda} \\ 0 = L_z &= -1 - 2\lambda x &&\implies z = -\frac{1}{2\lambda} \\ 0 = L_\lambda &= x^2 + y^2 + z^2 - 1 &&\implies 3\Big(\frac{1}{2\lambda}\Big)^2 - 1 = 0 \qquad \implies \lambda = \pm\frac{\sqrt{3}}{2} \end{aligned}$$

The critical points are $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, where $f=-\sqrt{3}$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, where $f=\sqrt{3}$. So, the max is $f=\sqrt{3}$ and the min is $f=-\sqrt{3}$.

4. Find a, b and c so that the volume $4\pi abc/3$ of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passing through the point $(1,2,1)$ is as small as possible.

SOLUTION: Define $L(a,b,c,\lambda)=\frac{4}{3}\pi abc-\lambda(\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1)$. Then

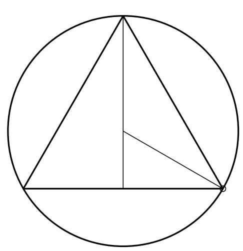
$$\begin{aligned} 0=L_a &= \frac{4}{3}\pi bc+\frac{2\lambda}{a^3} &&\implies \frac{3}{2\pi}\lambda=-a^3bc \\ 0=L_b &= \frac{4}{3}\pi ac+\frac{8\lambda}{b^3} &&\implies \frac{3}{2\pi}\lambda=-\frac{1}{4}ab^3c \\ 0=L_c &= \frac{4}{3}\pi ab+\frac{2\lambda}{c^3} &&\implies \frac{3}{2\pi}\lambda=-abc^3 \\ 0=L_\lambda &= \frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1 \end{aligned}$$

The equations $-\frac{3}{2\pi}\lambda=a^3bc=\frac{1}{4}ab^3c$ force $b=2a$ (since we want $a,b,c>0$). The equations $-\frac{3}{2\pi}\lambda=a^3bc=abc^3$ force $a=c$. Hence

$$0=\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1=\frac{3}{a^2}-1 \implies a=c=\sqrt{3}, \quad b=2\sqrt{3}$$

5. Find the triangle of largest area that can be inscribed in the circle $x^2+y^2=1$.

SOLUTION: Inscribe the base of the triangle and choose a coordinate system in which the base is horizontal. Pick the vertex of the triangle. For a given base, the triangle has maximum height (and hence area) if the vertex is chosen to be at the "top" of the circle, as shown.



We are to mazimize $A=bh/2$ subject to $(h-1)^2+\left(\frac{b}{2}\right)^2=1$. Define

$$L(b,h,\lambda)=(1/2)bh-\lambda\big((h-1)^2+\left(\frac{b}{2}\right)^2-1\big)$$

Then

$$
\begin{aligned}
0 = L_b &= (1/2)h - (1/2)\lambda b \implies \frac{b^2}{4} = h(h-1) \\
0 = L_h &= (1/2)b - 2\lambda(h-1) \implies \lambda = \frac{b}{4(h-1)} \\
0 = L_\lambda &= -(h-1)^2 - \left(\frac{b}{2}\right)^2 + 1 \implies (h-1)^2 + h(h-1) = 1 \\
&\implies 2h^2 - 3h = 0
\end{aligned}
$$

So h must be either 0 (which cannot give maximum area) or $h = \frac{3}{2}$ and $b = \sqrt{3}$. All three sides of the triangle have length $\sqrt{3}$, so the triangle is equilateral (surprise!).

6. For each of the following, evaluate the given double integral **without** using iteration. Instead, interpret the integral as an area or some other physical quantity.

   (a) $\iint_R dx\, dy$ where R is the rectangle $-1 \le x \le 3,\ -4 \le y \le 1$.

   (b) $\iint_D (x+3) dx\, dy$, where D is the half disc $0 \le y \le \sqrt{4 - x^2}$.

   (c) $\iint_R (x+y) dx\, dy$ where R is the rectangle $0 \le x \le a,\ 0 \le y \le b$.

   (d) $\iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$ where R is the region $x^2 + y^2 \le a^2$.

   (e) $\iint_R \sqrt{b^2 - y^2}\, dx\, dy$ where R is the rectangle $0 \le x \le a,\ 0 \le y \le b$.

SOLUTION:

(a) $\iint_R dx\, dy$ is the area of a rectangle with sides of lengths 4 and 5. This area is $\iint_R dx\, dy = 4 \times 5 = 20$.

(b) $\iint_D x\ dx\, dy = 0$ because x is odd under reflection about the y–axis, while the domain of integration is symmetric about the y–axis. $\iint_D 3\ dx\, dy$ is the three times the area of a half disc of radius 2. So, $\iint_D (x+3) dx\, dy = 3 \times (1/2) \times \pi 2^2 = 6\pi$.

(c) $\iint_R x\ dx\, dy / \iint_R dx\, dy$ is the average value of x in the rectangle R, namely $\frac{a}{2}$. Similarly, $\iint_R y\ dx\, dy / \iint_R dx\, dy$ is the average value of y in the rectangle R, namely $\frac{b}{2}$. $\iint_R dx\, dy$ is area of the rectangle R, namely ab. So, $\iint_S (x+y) dx\, dy = (1/2)ab(a+b)$.

(d) $\iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$ is the volume of the region, V, with $0 \le z \le \sqrt{a^2 - x^2 - y^2}$, $x^2 + y^2 \le a^2$. This is the top half of a sphere of radius a. So, $\iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \frac{2}{3}\pi a^3$.

(e) e) $\iint_R \sqrt{b^2 - y^2}\, dx\, dy$ is the volume of the region, V, with $0 \le z \le \sqrt{b^2 - y^2}$, $0 \le x \le a$, $0 \le y \le b$. $y^2 + z^2 \le b^2$ is a cylinder of radius b centered on the x axis. $y^2 + z^2 \le b^2,\ y \ge 0,\ z \ge 0$ is one quarter of this cylinder. It has cross–sectional area $\frac{1}{4}\pi b^2$. V is the part of this quarter–cylinder with $0 \le x \le a$. It has length a and cross–sectional area $\frac{1}{4}\pi b^2$. So, $\iint_R \sqrt{b^2 - y^2}\, dx\, dy = \frac{1}{4}\pi a b^2$.

7. For each iterated integral, sketch the domain of integration and evaluate:

(a)

$$I = \int_0^1 \int_y^1 e^{-x^2}\, dxdy$$

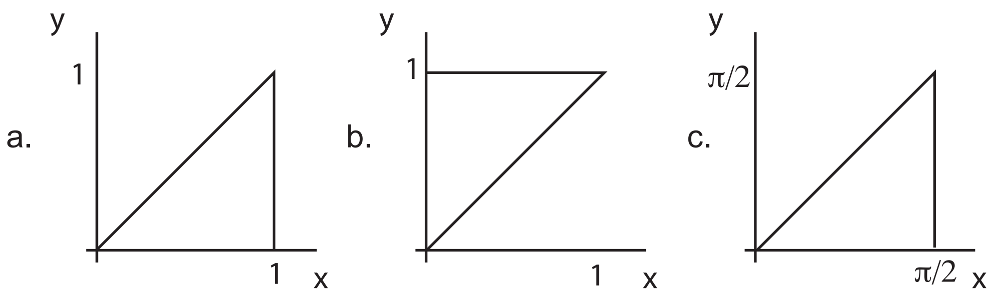
(b)

$$I = \int_0^1 \int_x^1 \frac{y^p}{x^2+y^2}\, dydx \;(p > 0)$$

(c)

$$I = \int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin x}{x}\, dxdy$$

SOLUTION: In each problem, the trick is to reverse the order of integration.



(a)

$$I = \int_0^1 \int_y^1 e^{-x^2}\, dxdy = \int_0^1 \int_0^x e^{-x^2}\, dydx = \int_0^1 \left[ye^{-x^2}\right]_{y=0}^{y=x} dx$$
$$= \int_0^1 xe^{-x^2} dx = \left[-\frac{e^{-x^2}}{2}\right]_0^1 = \frac{1-e^{-1}}{2}$$

(b)

$$I = \int_0^1 \int_x^1 \frac{y^p}{x^2+y^2}\, dydx = \int_0^1 \int_0^y \frac{y^p}{x^2+y^2}\, dxdy = \int_0^1 y^p \int_0^y \frac{1}{x^2+y^2}\, dxdy$$
$$= \int_0^1 y^p \left[\frac{1}{y}\tan^{-1}(x/y)\right]_{x=0}^{x=y} dy = \int_0^1 y^{p-1}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) dy = \int_0^1 \frac{\pi}{4} y^{p-1} dy$$
$$= \frac{\pi}{4}\left[\frac{y^p}{p}\right]_0^1 = \frac{\pi}{4p}$$

(c)

$$I = \int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin x}{x}\, dxdy = \int_0^{\pi/2} \int_0^x \frac{\sin x}{x}\, dydx = \int_0^{\pi/2} \sin x\, dx = 1$$