Math 263 Assignment 9 - Solutions

1. Find the flux of $\vec{F} = (x^2 + y^2)\vec{k}$ through the disk of radius 3 centred at the origin in the xy plane and oriented upward.

   Solution The unit normal vector to the surface is $\vec{n} = \vec{k}$. The flux is thus given by:

$$\int\int_S \vec{F}.dS = \int\int_S \vec{F}.\vec{n}\,dS = \int\int_S x^2 + y^2 dS$$
$$= \int_0^{2\pi}\int_0^3 r^2\, r dr d\theta = 2\pi\frac{3^4}{4} = \frac{81\pi}{2}$$

2. For each of these situations, (i) Sketch S, (ii) Parametrize S, (iii) find the vector and scalar elements $d\vec{S}$ and dS for your parametrization, (iv) calculate the indicated surface or flux integral.

   (a) S given by $z = x^2y^2$, $-1 \le x \le 1$, $-1 \le y \le 1$ oriented positive up. Calculate $\int\int_S \vec{F}.d\vec{S}$ for $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$.

   (b) S is the surface of $4x^2 + 4y^2 + z^2 - 6z + 5 = 0$ oriented inward. Calculate the surface area of S.

   (c) S is the surface of intersection of the sphere $x^2 + y^2 + z^2 \le 4$ and the plane $z = 1$ oriented away from the origin. Calculate the flux through the surface of the electrical field $\vec{E}(\vec{r}) = \frac{\vec{r}}{|\vec{r}|^3}$.

   Solution

   (a) We parameterize S by $\vec{r}(x, y) = x\vec{i} + y\vec{j} + x^2y^2\vec{k}$ over $-1 \le x \le 1$, $-1 \le y \le 1$. The vector area element is given by

$$d\vec{S} = \pm(\vec{r}_x \times \vec{r}_y)\, dx\, dy = \pm\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2xy^2 \\ 0 & 1 & 2x^2y \end{vmatrix} dx\, dy = \pm\begin{pmatrix} -2xy^2 \\ -2x^2y \\ 1 \end{pmatrix} dx\, dy$$

   Since we want this to be oriented upwards, we have to pick the plus option. The scalar area element is $dS = \sqrt{4x^2y^4 + 4x^4y^2 + 1}dx\, dy$.

$$\int\int_S \vec{F}.d\vec{S} = \int\int_D \begin{pmatrix} x \\ 1 \\ x^2y^2 \end{pmatrix} \cdot \begin{pmatrix} -2xy^2 \\ -2x^2y \\ 1 \end{pmatrix} dA = \int_{-1}^{1}\int_{-1}^{1} x^2(-y^2 - 2y)dx\, dy = \ldots = -\frac{4}{9}$$

   (b) Completing the square gives $4x^2 + 4y^2 + (z - 3)^2 = 4$ so S is an ellipsoid centred at $(0, 0, 3)$. In cylindrical coordinates, S consists of the points (r, θ, z) where $0 \le \theta \le 2\pi$, $1 \le z \le 5$, and $4r^2 + (z - 3)^2 = 4$, or equivalently, $r = \frac{1}{2}\sqrt{4 - (z - 3)^2}$. Therefore, we can parametrize the surface using θ and z by

$$\vec{r}(\theta, z) = \begin{pmatrix} \frac{1}{2}\cos\theta\sqrt{4 - (z - 3)^2} \\ \frac{1}{2}\sin\theta\sqrt{4 - (z - 3)^2} \\ z \end{pmatrix}.$$

The vector area element is given by

$$d\vec{S} = \pm\left(\vec{r}_\theta \times \vec{r}_z\right) d\theta dz \quad = \quad \pm \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{2}\sin\theta\sqrt{4-(z-3)^2} & \frac{1}{2}\cos\theta\sqrt{4-(z-3)^2} & 0 \\ -\frac{1}{2}\cos\theta\frac{z-3}{\sqrt{4-(z-3)^2}} & -\frac{1}{2}\sin\theta\frac{z-3}{\sqrt{4-(z-3)^2}} & 1 \end{pmatrix} d\theta dz$$

$$= \quad \pm \begin{pmatrix} \frac{1}{2}\cos\theta\frac{z-3}{\sqrt{4-(z-3)^2}} \\ \frac{1}{2}\sin\theta\frac{z-3}{\sqrt{4-(z-3)^2}} \\ \frac{1}{4}(z-3) \end{pmatrix}$$

We want the orientation inward, so we have to pick the version that, say, gives us downward orientation at the upper tip of the ellipse $(0,0,5)$, thus we pick the negative sign.

The scalar area element is

$$dS = |d\vec{S}| = \frac{1}{4}\sqrt{-3z^2+18z-11}r^2 dr\, d\theta$$

and therefore the surface area is just the integral of this over the parameterization,

$$\begin{aligned} A(S) &= \int\int_S 1\, dS = \int_0^{2\pi}\int_1^5 \left[\frac{1}{4}\sqrt{-3z^2+18z-11}\right] dz\, d\theta \\ &= 2\pi\frac{1}{4}\int_1^5 \sqrt{16-3(z-3)^2}dz. \end{aligned}$$

Now do the substitution $u = \sqrt{3}(z-3)$:

$$\begin{aligned} A(S) &= \frac{\pi}{2}\int_{-2\sqrt{3}}^{2\sqrt{3}} \sqrt{16-u^2}\frac{du}{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}\left[\frac{1}{2}u\sqrt{16-u^2} + 8\sin^{-1}\frac{u}{4}\right]_{-2\sqrt{3}}^{2\sqrt{3}} \\ &= \frac{\pi}{2\sqrt{3}}\left[\left(2\sqrt{3}+\frac{8\pi}{3}\right) - \left(-2\sqrt{3}-\frac{8\pi}{3}\right)\right] = \frac{\pi}{2\sqrt{3}}\left(4\sqrt{3}+\frac{16\pi}{3}\right) \\ &= 2\pi\left(1+\frac{4\pi}{3\sqrt{3}}\right). \end{aligned}$$

(c) The surface is a disk of radius $\sqrt{3}$ centred at $(0,0,1)$ and lying in the plane $z=1$. The easiest parameterization is in the cylindrical coordinates (r,θ,z) but with $z=1$:

$$\vec{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1 \end{pmatrix}, \; 0 \le r \le \sqrt{3},\; 0 \le \theta \le 2\pi.$$

The vector area element is

$$d\vec{S} = \pm(\vec{r}_r \times \vec{r}_\theta) dr\, d\theta = \pm \begin{vmatrix} \vec{i} & \vec{j} & \vec{j} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\, d\theta = \pm \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\, d\theta.$$

The scalar area element is

$$dS = |d\vec{S}| = r^2 dr\, d\theta$$

Finally, the flux through the surface is

$$\begin{aligned}\int\int_S \vec{E}.d\vec{S} &= \int_0^{\sqrt{3}}\int_0^{2\pi} \frac{1}{(r^2+1)^{3/2}} \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\, d\theta = 2\pi \int_0^{\sqrt{3}} \frac{r}{(r^2+1)^{3/2}} dr \\ &= 2\pi \left[\frac{-1}{(r^2+1)^{1/2}}\right]_0^{\sqrt{3}} = \pi\end{aligned}$$

3. For constants a, b, c, m, consider the vector field

$$\vec{F} = (ax + by + 5z)\vec{i} + (x + cz)\vec{j} + (3y + mx)\vec{k}.$$

(a) Suppose that the flux of $\vec{F}$ through any closed surface is 0. What does this tell you about the value of the constants a, b, c and m?

(b) Suppose instead that the line integral of $\vec{F}$ around any closed curve is 0. What does this tell you about the values of the constants a, b, c and m?

Solution

(a) If the flux of $\vec{F}$ through any closed surface is 0, then by the divergence theorem, the vector field must have zero divergence.

$$\vec{\nabla}.\vec{F} = a = 0$$

This tells us that $a = 0$ but it does not tell us anything about b, c or m.

(b) If the line integral of $\vec{F}$ around any closed curve is 0, this means that the vector field has curl equal to zero everywhere.

$$\vec{\nabla} \times \vec{F} = (3 - c)\vec{i} + (5 - m)\vec{j} + (1 - b)\vec{k}$$

This tells us that $c = 3, m = 5$ and $b = 1$. It does not tell us anything about a.

4. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Consider the vector field

$$\vec{E} = \frac{\vec{r}}{|\vec{r}|^3}.$$

Find $\int_S \vec{E}.d\vec{A}$ where S is the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$. Give reasons for your calculation.

Solution The divergence of $\vec{E}$ is zero (check it!). However, the divergence theorem does not apply because $\vec{E}$ is not defined at $(0, 0, 0)$. To get around this, we can define the sphere B by $x^2 + y^2 + z^2 \leq a^2$ for some small a, with normal vector oriented towards $(0, 0, 0)$ and apply the divergence theorem to the region R in between S and B:

$$\begin{aligned}\int\int_S \vec{E}.dS + \int\int_B \vec{E}.dS &= \int\int\int_R \vec{\nabla}.\vec{E}\, dV = 0 \\ \int\int_S \vec{E}.dS &= -\int\int_B \vec{E}.dS\end{aligned}$$

The integral over B is easy to do. The inward-facing unit normal vector to B is

$$\vec{n} = -\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = -\frac{\vec{r}}{|\vec{r}|}$$

and so the surface integral is

$$\begin{aligned} \int\int_S \vec{E}.dS &= -\int\int_B \vec{E}.dS = +\int\int_B \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|}\, dS \\ &= \int\int_B \frac{\vec{r}.\vec{r}}{|\vec{r}|^4}\, dS = \int\int_B \frac{1}{|\vec{r}|^2}\, dS \end{aligned}$$

On B, $|\vec{r}| = a$.

$$\begin{aligned} \int\int_S \vec{E}.dS &= \int\int_B \frac{1}{|\vec{r}|^2}\, dS = \int\int_B \frac{1}{a^2}\, dS = \frac{1}{a^2} \int\int_B 1\, dS \\ &= \frac{1}{a^2} 4\pi a^2 = 4\pi. \end{aligned}$$

5. Use geometric reasoning to find $I = \int\int_S \vec{F}.d\vec{S}$ by inspection for the following three situations. Explain your answers. In each case, a and b are positive constants.

   (a) $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ and S is the surface consisting of three squares with one corner at the origin and positive sides facing the first octant. The squares have sides ($b\vec{i}$ and $b\vec{j}$), ($b\vec{j}$ and $b\vec{k}$), and ($b\vec{i}$ and $b\vec{k}$), respectively.

   (b) $\vec{F}(x, y, z) = (x\vec{i} + y\vec{j}) \ln(x^2 + y^2)$, and S is the surface of the cylinder (including top and bottom) where $x^2 + y^2 \leq a^2$ and $0 \leq z \leq b$.

   (c) $\vec{F}(x, y, z) = (x\vec{i} + y\vec{j} + z\vec{k})e^{-(x^2+y^2+z^2)}$, and S is the spherical surface $z^2 + y^2 + z^2 = a^2$.

Solution

(a) The square with sides $b\vec{i}$ and $b\vec{j}$ has normal $\hat{N} = \vec{k}$ and lies in the plane where $z = 0$. Thus $\vec{F} \cdot \vec{N} = 0$ on this part of the surface. The same thing happens on the other two squares so we have that the whole flux integral is zero.

(b) On the flat top of the cylinder, the outward normal is $\vec{N} = \vec{k}$, and we have $\vec{F} \cdot \vec{N} = 0$ on this part of the surface. The same thing happens on the bottom. On the sides, the outward unit normal at (x, y, z) is $\vec{N} = (\frac{x}{a}, \frac{y}{a}, 0)$, so we have

$$\vec{F} \cdot \vec{N} = \left(x \left(\frac{x}{a} \right) + y \left(\frac{y}{a} \right) \right) \ln(x^2 + y^2) = a \ln(a^2) = 2a \ln a.$$

It follows that

$$\int\int_S \vec{F} \cdot \vec{N}\, dS = 2a \ln a\, A(\text{cylinder} - \text{side}) = 2a \ln a[2\pi ab] = 4\pi a^2 b \ln a.$$

(c) On the surface of the sphere, the outward-facing unit normal vector is

$$\vec{n} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = -\frac{\vec{r}}{|\vec{r}|}$$

Hence

$$\begin{aligned} \vec{F} \cdot \vec{N} &= \vec{r}e^{-a^2} \cdot \frac{\vec{r}}{a} = ae^{-a^2}. \\ \int\int_S \vec{F} \cdot \vec{N}\, dS &= ae^{-a^2} \int\int_S 1\, dS = ae^{-a^2}[4\pi a^2] = 4\pi a^3 e^{-a^2}. \end{aligned}$$

$\vec{F}(x, y, z) = (x\vec{i} + y\vec{j} + z\vec{k})e^{-(x^2+y^2+z^2)}$, and S is the spherical surface $z^2 + y^2 + z^2 = a^2$.

6. Let S be the boundary surface of the solid given by $0 \le z \le \sqrt{4 - y^2}$ and $0 \le x \le \frac{\pi}{2}$.

   (a) Find the outward unit normal vector field $\vec{N}$ on each of the four sides of S.

   (b) Find the total outward flux of $\vec{F} = 4 \sin x\vec{i} + z^3\vec{j} + yz^2\vec{k}$ through S.

   Do the calculation directly (don't use the Divergence theorem).

   Solution

   (a) On the surface $z = 0$ (the bottom), $\vec{N} = -\vec{k}$. On the side $x = 0$, $\vec{N} = -\vec{i}$. On the side $\vec{x} = \pi/2$, $\vec{N} = \vec{i}$. On the top surface ($z = \sqrt{4 - y^2}$), we have to calculate. Parameterizing this surface as $\vec{r}(x, y) = x\vec{i} + y\vec{j} + \sqrt{4 - y^2}\vec{k}$, we can use

$$\vec{N} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{1}{|\vec{r}_x \times \vec{r}_y|} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & \frac{-y}{\sqrt{4-y^2}} \end{vmatrix} = \frac{\sqrt{4 - y^2}}{2} \begin{pmatrix} 0 \\ \frac{y}{\sqrt{4-y^2}} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ y \\ \sqrt{4 - y^2} \end{pmatrix}$$

   Note that this has the correct (upward) orientation.

   (b) Now we have to integrate over each surface in turn and add them up.

      i. On the bottom surface S_1, we have $z = 0$, $\vec{N} = -\vec{k}$.

$$\int\int_{S_1} \vec{F}.dS = \int\int_{S_1} \begin{pmatrix} 4 \sin x \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS = 0.$$

      ii. On the left surface S_2, we have $x = 0$, $\vec{N} = -\vec{i}$.

$$\int\int_{S_2} \vec{F}.dS = \int\int_{S_2} \begin{pmatrix} 0 \\ z^3 \\ yz^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dS = 0.$$

iii. On the right surface S_3, we have $x = \pi/2$, $\vec{N} = \vec{i}$.

$$\int\int_{S_3} \vec{F}.dS = \int\int_{S_2} \begin{pmatrix} 4 \\ z^3 \\ yz^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dS = 4A(S_3).$$

7. Evaluate, both by direct integration and by Stokes' Theorem, $\int_C (z\,dx + x\,dy + y\,dz)$ where C is the circle $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$. Orient C so that its projection on the xy–plane is counterclockwise.

Solution *By direct integration*: We need a parameterization of C. C is the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 1$. The projection of C on the xy-plane is $x^2 + y^2 + (-x - y)^2 = 1$ or $2x^2 + 2xy + 2y^2 = 1$ or $\frac{3}{2}(x+y)^2 + \frac{1}{2}(x-y)^2 = 1$. This looks a bit like an ellipse, so we can use $x + y = \sqrt{\frac{2}{3}} \cos\omega, x - y = -\sqrt{2}\sin\omega$, for instance. (The minus sign in the expression for $x - y$ is chosen so that the motion is counterclockwise.) Solving these equations for x and y, and using $z = -x - y$, we get

$$\vec{r}(\theta) = \begin{pmatrix} \frac{1}{\sqrt{6}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{\sqrt{6}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \\ \frac{-2}{\sqrt{6}}\cos\theta \end{pmatrix} \qquad \text{and} \qquad \vec{r'}(\theta) = \begin{pmatrix} -\frac{1}{\sqrt{6}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta \\ -\frac{1}{\sqrt{6}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta \\ \frac{2}{\sqrt{6}}\sin\theta \end{pmatrix}.$$

Now

$$\begin{aligned}
&\int_C (z\,dx + x\,dy + y\,dz) = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)).\vec{r'}(\theta)d\theta \\
= \quad &\int_0^{2\pi} \left[\frac{-2}{\sqrt{6}}\cos\theta\left(-\frac{1}{\sqrt{6}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right) + \left(\frac{1}{\sqrt{6}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\left(-\frac{1}{\sqrt{6}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)\right. \\
&\left. + \left(\frac{1}{\sqrt{6}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{2}{\sqrt{6}}\sin\theta\right)\right] d\theta \\
&= \int_0^{2\pi} \left[\frac{3}{\sqrt{12}}\cos^2\theta + \frac{3}{\sqrt{12}}\sin^2\theta + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{2} + \frac{1}{3}\right)\sin\theta\cos\theta\right] d\theta \\
&= \sqrt{3}\,\pi
\end{aligned}$$

Using Stokes theorem: To apply Stokes, we need a surface S that has C as its boundary. Choose S to be the portion of the plane $x + y + z = 0$ interior to the sphere. The unit normal to S is then $\hat{n} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$. Also $\text{curl}\vec{F} = \vec{i} + \vec{j} + \vec{k}$ so, applying Stokes theorem,

$$\int_C \vec{F}.d\vec{r} = \int\int_S \text{curl}\vec{F}.\hat{n}\,dS = \int\int_S \frac{3}{\sqrt{3}}\,dS = \sqrt{3}A(S) = \sqrt{3}\,\pi.$$

8. Evaluate $\int_C (x\sin y^2 - y^2)dx + (x^2 y\cos y^2 + 3x)dy$ where C is the counterclockwise boundary of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$ and $(0, 2)$.

Solution Use Green's theorem to convert the line integral to an area integral.

$$\int_C (x\sin y^2 - y^2)dx + (x^2 y\cos y^2 + 3x)dy = \int\int_{\text{trapezoid}} \left(2xy\cos y^2 + 3 - 2xy\cos y^2 - 2y\right) dA$$

$$= \int\int_{\text{trapezoid}} (3-2y)\, dA = 3A(\text{trapezoid}) = 9$$

The integral of $2y$ is zero by symmetry (draw a picture of the trapezoid to see this).

9. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = ye^x\vec{i} + (x+e^x)\vec{j} + z^2\vec{k}$ and C is the curve

$$\vec{r}(t) = (1+\cos t)\vec{i} + (1+\sin t)\vec{j} + (1-\sin t - \cos t)\vec{k}$$

Solution First check if $\vec{F}$ is conservative by taking the curl.

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ ye^x & (x+e^x) & z^2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1+e^x-e^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0.$$

$\vec{F}$ is not conservative, but the curl is simple. This suggests using Stokes theorem. To use Stokes, we have to come up with a surface that has C as its boundary. The projection of C in the x-y plane is just a circle of radius one, centred at (1,1). We can write $x = 1+\cos t$ and $y = 1+\sin t$ and then $z = 1-(y-1)-(x-1) = 3-x-y$ on the curve. The simplest way to come up with the surface, then, is to parameterize it as $\vec{r}(x,y) = x\vec{i} + y\vec{j} + (3-x-y)\vec{k}$, taken over the disk of radius one centred at $(1,1)$. Let's call this surface S. By Stokes' theorem,

$$\begin{aligned} \int_C \vec{F}.d\vec{r} &= \int\int_S \text{curl}\vec{F}.dS = \int\int_{\text{disk}} \vec{k}.(\vec{r}_x \times \vec{r}_y)dA = \int\int_{\text{disk}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} dA \\ &= \int\int_{\text{disk}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} dA = \int\int_{\text{disk}} 1\, dA = A(\text{disk}) = \pi. \end{aligned}$$