Math 263 Practice Problem Set 1 Solutions

1. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2+y^2$ and the ellipsoid $4x^2+y^2+z^2 = 9$ at the point $(-1, 1, 2)$.

   Solution. Let the parametric representation of the ellipse be given by $(x, y(x), z(x))$, where $y(x)$ and $z(x)$ are given by the two equations

$$z(x) = x^2 + y^2, \quad 4x^2 + y(x)^2 + z(x)^2 = 9. \tag{1}$$

   The tangent vector at the point $(-1, 1, 2)$ is given by $(1, y'(-1), z'(-1))$. For this we differentiate the equations in (1) with respect x; this gives

$$z'(x) = 2x + 2yy'(x), \qquad 8x + 2y(x)y'(x) + 2z(x)z'(x) = 0.$$

   Plugging in $x = -1$, $y(x) = 1$, $z = 2$, we obtain

$$2z'(-1) - 4y'(-1) = -4, \qquad 2z'(-1) + y'(-1) = 4.$$

   Solving the two linear equations give $y'(-1) = \frac{8}{5}$, $z'(-1) = \frac{6}{5}$. The direction of the tangent vector is therefore $(1, \frac{8}{5}, \frac{6}{5})$, or equivalently $(5, 8, 6)$. The equation of the tangent line is therefore

$$x + 1 = 5t, \quad y - 1 = 8t, \quad z - 2 = 6t.$$

   □

2. Show that the sum of the x, y and z-intercepts of any tangent plane of the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

   Solution. Let us fix any point (x_0, y_0, z_0) in the first octant. Let us differentiate the equation of the surface with respect to x and y to solve for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{z}}\frac{\partial z}{\partial x} = 0, \qquad \frac{1}{2\sqrt{y}} + \frac{1}{2\sqrt{z}}\frac{\partial z}{\partial y} = 0.$$

   Therefore $\frac{\partial z}{\partial x} = -\frac{\sqrt{z_0}}{\sqrt{x_0}}$, $\frac{\partial z}{\partial y} = -\frac{\sqrt{z_0}}{\sqrt{y_0}}$. The normal direction to the surface at the point (x_0, y_0, z_0) is therefore $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1)(x_0, y_0) = -(\sqrt{\frac{z_0}{x_0}}, \sqrt{\frac{z_0}{y_0}}, 1)$. The equation of the tangent plane at (x_0, y_0, z_0) is then

$$\sqrt{\frac{z_0}{x_0}}(x - x_0) + \sqrt{\frac{z_0}{y_0}}(y - y_0) + (z - z_0) = 0,$$

$$\text{or,} \quad \frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}.$$

The lengths of the x, y and z-intercepts of the tangent plane at (x_0, y_0, z_0) are therefore $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$, and the sum of the intercepts is

$$\sqrt{c}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = c,$$

which is independent of (x_0, y_0, z_0). □

3. The radius of a right circular cone is increasing at a rate 1.8 in/s, while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 inches and the height is 140 inches?

Solution. The volume of a right circular cone is

$$V(r, h) = \frac{1}{3}\pi r^2 h.$$

Therefore

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{3}\pi \left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right] \\ &= \frac{\pi}{3}\left[240(140)(1.8) - (120)^2(2.5)\right] = 8160\pi \text{in}^3/s. \end{aligned}$$

□

4. Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

Solution. We first compute the gradient vector:

$$\nabla f(x, y) = \langle -y^2e^{-xy}, e^{-xy} - xye^{-xy}\rangle, \qquad \text{hence} \qquad \nabla f(0, 2) = -4\mathbf{i} + \mathbf{j}.$$

Let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be the unit vector in the direction to be determined. Then $D_{\mathbf{u}}f(0, 2) = -4\cos\theta + \sin\theta = 1$. In order to solve this equation for θ, use the half-angle formulas:

$$\begin{aligned} &-4(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}, \\ &\text{or, } 3\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 5 = 0, \\ &\text{or, } \tan\frac{\theta}{2} = 1, -\frac{5}{3}. \end{aligned}$$

$$\text{Thus, } \frac{\theta}{2} = \frac{\pi}{4} \text{ and } \tan^{-1}(-\frac{5}{3}).$$

Check that $2\tan^{-1}(-5/3) = 2\pi - \cos^{-1}(8/17)$. □

5. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200+0.02x^2-0.001y^3$ where x, y, z are in meters. A fisherman starts at $(80, 60)$ and moves towards the buoy which is located $(0, 0)$. Is the boat getting deeper or shallower when he departs?

   (**Answer:** depth is increasing in the direction toward the buoy)

   Solution. The direction of movement of the boat is $\mathbf{u} = -\frac{1}{5}(4\mathbf{i}+3\mathbf{j})$. The gradient vector of $z = f(x, y)$ at (80, 60) is

$$0.04(80)\mathbf{i} - 0.003(60)^2\mathbf{j} = 3.2\mathbf{i} - 10.8\mathbf{j}.$$

   Therefore $D_{\mathbf{u}}f(80, 60) = -\frac{1}{5}(12.8 - 32.4) > 0$; in other words, the depth is increasing as the boat departs. ☐