Math 440/508, Fall 2008, Final Exam (Due date: Monday December 8)

Instructions

- Please drop off your final exam in the instructor's office by 5pm on Monday December 8. Slide your paper under the door if the office is closed.
- Unlike homework assignments, you must work on the final on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

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1. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{R}$, then f is entire.
2. Given two sequences of complex numbers $\{a_k : k \geq 0\}$ and $\{b_k : k \geq 0\}$, with $|a_k| \to \infty$, it is always possible to find an entire function F that satisfies $F(a_k) = b_k$ for all k. Prove this result in the following two steps.
   (a) First deal with the "finite" case: namely, given n distinct complex numbers $a_1, \cdots, a_n$ and another n complex numbers $b_1, \cdots, b_n$, construct a polynomial P of degree $\leq n-1$ with
   $$P(a_i) = b_i \quad \text{for } i = 1, \cdots, n.$$
   (b) Now let $\{a_k : k \geq 0\}$ be a sequence of distinct complex numbers such that $a_0 = 0$ and $|a_k| \to \infty$, and $E(z)$ denote a Weierstrass product associated with $\{a_k\}$. Given complex numbers $\{b_k : k \geq 0\}$, show that there exist integers $m_k \geq 1$ such that the series
   $$F(z) = \frac{b_0}{E'(0)} \frac{E(z)}{z} + \sum_{k=1}^{\infty} \frac{b_k}{E'(a_k)} \frac{E(z)}{z - a_k} \left(\frac{z}{a_k} \right)^{m_k}$$
   defines an entire function that satisfies
   $$F(a_k) = b_k \quad \text{for all } k \geq 0.$$
   The formula above is known as the Pringsheim interpolation formula.
3. Does there exist a holomorphic surjection from the unit disk to $\mathbb{C}$?
4. Prove that the function u defined by
   $$u(z) = \operatorname{Re} \left(\frac{i+z}{i-z} \right), \quad u(0,1) = 0$$
   is harmonic in the unit disk and vanishes on its boundary. Why does this not contradict the harmonic extension thoerem (as stated for instance on page 277, Gamelin)?