Homework 4 – Math 440/508

Due on Friday November 28

Problem 1. Suppose $0 < a_1 < a_2 < \cdots$. Find necessary and sufficient conditions on $\{a_n\}$ for the infinite product

$$\prod_{n=1}^{\infty} \frac{a_n - z}{a_n + z}$$

to converge on $\{z : \mathrm{Re}(z) > 0\}$.
[*Clarification added on November 15: An infinite product* $\prod_{n=1}^{\infty} f_n$ *of analytic functions "converges on* G*" if it converges in the metric toppology on* $\mathbb{H}(G)$ *induced by* ρ*, i.e., if it converges uniformly on compact subsets of* G*.*]

Problem 2. Find a nontrivial entire function f such that $f(m+in) = 0$ for all possible integers m, n. Find the most elementary solution possible.

Problem 3.

(a) Let $0 < |a| < 1$ and $|z| \leq r < 1$; show that

$$\left| \frac{a + |a|z}{(1 - \bar{a}z)} \right| \leq \frac{1+r}{1-r}.$$

(b) Let $\{a_n\} \subseteq \mathbb{C}$ be a sequence with $0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left[\frac{a_n - z}{1 - \bar{a}_n z} \right]$$

converges in $\mathbb{H}(B(0;1))$ and that $|B(z)| \leq 1$. What are the zeros of B? ($B(z)$ is called a *Blaschke product.*)

(c) Find a sequence $\{a_n\}$ in $B(0;1)$ such that $\sum_n (1 - |a_n|) < \infty$ and every point on the unit circle is a limit point of $\{a_n\}$.

Problem 4. There exists an entire function F which is "universal" in the sense that it approximates *every* entire function h. (Does this seem surprising?) More precisely, given any entire function h, there is an increasing sequence $\{N_k : k \geq 1\}$ of positive integers such that

$$\lim_{k \to \infty} F(z + N_k) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$. Note that the sequence $\{N_k\}$ depends only on h and works for every compact subset of $\mathbb{C}$.

Establish the following statements to deduce the existence of such a function F.

(a) Let $p_1, p_2, \cdots$ denote an enumeration of the collection of polynomials whose coefficients have rational real and imaginary parts. Show that it suffices to find an entire function F and an increasing sequence $\{M_n\}$ of positive integers such that

$$|F(z) - p_n(z - M_n)| < \frac{1}{n} \quad \text{whenever } z \in B(M_n; n).$$

Hint: Given h *entire, there exists a sequence* $\{n_k\}$ *such that* $\lim_{k\to\infty} p_{n_k}(z) = h(z)$ *uniformly on every compact subset of* $\mathbb{C}$.

(b) Construct F as an infinite series

$$F(z) = \sum_{n=1}^{\infty} u_n(z), \qquad \text{where } u_n(z) = p_n(z - M_n)e^{-c_n(z-M_n)^2}.$$

Show that for a suitable choice of positive constants $c_n \to 0$ and $M_n \to \infty$, the condition mentioned in part (a) holds.

Problem 5. Let r_1, r_2, R_1, R_2 be positive numbers. Show that the two annuli $\mathrm{Ann}(0; r_i, R_i)$, $i = 1, 2$ are conformally equivalent via a map that extends continuously up to the boundary if and only if $r_1/R_1 = r_2/R_2$. (Note: The hypothesis of "continuous extension up to the boundary" is not needed and can be deduced from conformality. You may however assume this for the problem.)

Problem 6. Can there exist an analytic function $f : \mathrm{Ann}(0; 0, 1) \to B(0, 1)$ with $f'(z) \neq 0$ for all $z \in \mathrm{Ann}(0; 0, 1)$ such that $f(\mathrm{Ann}(0; 0, 1)) = B(0; 1)$? If yes, find such a function. If not, explain why not.

Problem 7. Given a domain $G \subseteq \mathbb{C}$, let $\{f_n\} \subseteq \mathbb{H}(G)$. If the sequence $\{\mathrm{Re}(f_n)\}$ converges uniformly on compact subsets of G and there exists z_0 for which $\{f_n(z_0)\}$ converges then prove that $\{f_n\}$ converges uniformly on compact subsets of G.

Problem 8. The aim of this problem is to guide you through a second proof of the Riemann mapping theorem, one that is due to Koebe. This proof is "constructive" in a weak sense – it specifies an iterative procedure that in the limit produces the conformal map required by the statement of the theorem. It does not invoke any result on normal families. However, the following result on harmonic functions is needed.

Harnack's Theorem. *Let* u_n *be a sequence of harmonic functions in a domain* G.

(i) If $u_n \to u$ *uniformly on compact subsets of* G*, then* u *is harmonic in* G.

(ii) If $u_1 \le u_2 \le u_3 \cdots$*, then either* $\{u_n\}$ *converges uniformly on compact subsets of* G *or* $u_n(z) \to \infty$ *for every* $z \in G$.

Let Ω be a proper simply connected domain in $\mathbb{C}$, $a \in \Omega$. Our goal is to obtain a conformal bijection f from Ω onto $\mathbb{D} = B(0;1)$ such that $f(a) = 0$ and $f'(a) > 0$. Assuming Harnack's theorem, fill in the details of the following outline of Koebe's proof.

(a) Prove that one can assume without loss of generality that $\Omega \subsetneq \mathbb{D}$, and $a = 0$.

(b) In the remainder of the problem we will construct domains $\Omega_1, \Omega_2, \cdots$ and conformal maps $f_1, f_2, \cdots$ such that $f_n(\Omega_{n-1}) = \Omega_n$ and so that the functions $f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1$ converge to the conformal mapping of Ω onto $\mathbb{D}$.

Set $\Omega_0 = \Omega$. Supposing Ω_{n-1} is constructed, let r_n denote the radius of the largest ball centered at the origin that is contained in Ω_{n-1}. Let α_n be a boundary point of Ω_{n-1} such that $|\alpha_n| = r_n$, choose β_n such that $\beta_n^2 = -\alpha_n$, and put

$$F_n = \varphi_{-\alpha_n} \circ s \circ \varphi_{-\beta_n}.$$

Here $\varphi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha} z)$, and $s(w) = w^2$. Show that F_n has a holomorphic inverse G_n.

(c) Pick $\lambda_n \in \mathbb{C}$, $|\lambda_n| = 1$ such that $f_n = \lambda_n G_n$ satisfies $f_n'(0) > 0$. Verify that

$$f_n'(0) = \frac{1 + r_n}{2\sqrt{r_n}} > 1.$$

(d) Put $\psi_0(z) = z$ and $\psi_n(z) = f_n(\psi_{n-1}(z))$. Show that ψ_n is an injective mapping of Ω onto a region $\Omega_n \subseteq \mathbb{D}$, that $\{\psi_n'(0)\}$ is bounded and

$$\psi_n'(0) = \prod_{k=1}^{n} \frac{1 + r_k}{2\sqrt{r_k}}.$$

Conclude that $r_n \to 1$ as $n \to \infty$.

(e) Write $\psi_n(z) = z h_n(z)$ for $z \in \Omega$, and show that $|h_n| \leq |h_{n+1}|$. Deduce using Harnack's theorem and Problem 7 that $\{\psi_n\}$ converges uniformly on compact subsets of Ω and show that $\lim_{n\to\infty} \psi_n$ is the conformal mapping specified by the Riemann Mapping Theorem.

(f) Let us now work out an example where we can see Koebe's algortihm in action. Starting with $\Omega_0 = \mathbb{D} \setminus [-\frac{1}{2}, -1)$, give an explicit description of the sets Ω_n.

Problem 9. Given an open set $G \subseteq \mathbb{C}$, let K_n denote the sequence of compact subsets of G given by

$$K_n = \{z : |z| \leq n\} \cap \left\{ z : d(z, \mathbb{C} \setminus G) \geq \frac{1}{n} \right\}.$$

Show that every component of $\mathbb{C}_\infty \setminus K_n$ contains some component of $\mathbb{C}_\infty \setminus G$.

Problem 10. Show that the topology induced on the unit disk by the Poincaré metric is the usual planar topology. Show also that the unit disk when equipped with the Poincaré metric is a complete metric space. Thus, a topological space may be equipped with two equivalent metrics, one of which is complete and the other is not. Does this seem counter-intuitive? Endow $\mathbb{R}$ with two metrics that generate the same Euclidean topology, one of which is complete and the other is not.

Problem 11. We have seen in class that any conformal map of the unit disk is an isometry of the pair $(\mathbb{D}, \rho)$, where ρ is the Poincaré metric. Now prove the following converse: If $\tilde{\rho}$ is a metric on $\mathbb{D}$ such that every conformal self-map of the disk is an isometry of the pair $(\mathbb{D}, \tilde{\rho})$, then $\tilde{\rho}$ is a constant multiple of the Poincaré metric ρ.