8. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ and

$$\iint_S y\,dS = \iint_D y\sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\,dA = \int_0^1 \int_0^1 y\sqrt{x+y+1}\,dx\,dy$$

$$= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1} dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2} - (y+1)^{3/2}\right] dy$$

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\iint_S y\,dS = \tfrac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \tfrac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt = \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2} - \tfrac{2}{5}t^{5/2}\right]_1^2$$

$$= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2} - 2^{7/2}) - \tfrac{4}{5}(3^{5/2} - 2^{5/2}) - \tfrac{2}{7}(2^{7/2} - 1) + \tfrac{2}{5}(2^{5/2} - 1)\right]$$

$$= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3} + \tfrac{8}{35}\sqrt{2} - \tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)$$

16. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 5$; and the back, S_3, in the plane $x = 0$.

On S_1: the surface is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\cos v\,\mathbf{j} + 3\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$, and $0 \le x \le 5 - y \;\Rightarrow$ $0 \le u \le 5 - 3\cos v$. Then $\mathbf{r}_u \times \mathbf{r}_v = -3\cos v\,\mathbf{j} - 3\sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{9\cos^2 v + 9\sin^2 v} = 3$, so

$$\iint_{S_1} xz\,dS = \int_0^{2\pi}\int_0^{5-3\cos v} u(3\sin v)(3)\,du\,dv = 9\int_0^{2\pi}\left[\tfrac{1}{2}u^2\right]_{u=0}^{u=5-3\cos v}\sin v\,dv$$

$$= \tfrac{9}{2}\int_0^{2\pi}(5-3\cos v)^2\sin v\,dv = \tfrac{9}{2}\left[\tfrac{1}{9}(5-3\cos v)^3\right]_0^{2\pi} = 0.$$

On S_2: $\mathbf{r}(y,z) = (5-y)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = |\mathbf{i} + \mathbf{j}| = \sqrt{2}$, where $y^2 + z^2 \le 9$ and

$$\iint_{S_2} xz\,dS = \iint_{y^2+z^2\le 9}(5-y)z\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^3 (5 - r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^3 (5r^2 - r^3\cos\theta)(\sin\theta)dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\tfrac{5}{3}r^3 - \tfrac{1}{4}r^4\cos\theta\right]_{r=0}^{r=3}\sin\theta\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(45 - \tfrac{81}{4}\cos\theta\right)\sin\theta\,d\theta = \sqrt{2}\left(\tfrac{4}{81}\right)\cdot\tfrac{1}{2}\left(45 - \tfrac{81}{4}\cos\theta\right)^2\Big]_0^{2\pi} = 0$$

On S_3: $x = 0$ so $\iint_{S_3} xz\,dS = 0$. Hence $\iint_S xz\,dS = 0 + 0 + 0 = 0$.

22. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the disk $\{(x,y) \mid x^2 + y^2 \le 1\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^4\right] dA = -\iint_D\left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^4\right] dA$$

$$= -\int_0^{2\pi}\int_0^1\left(\frac{-r^2}{r} + r^4\right) r\,dr\,d\theta = -\int_0^{2\pi} d\theta\int_0^1 (r^5 - r^2)\,dr = -2\pi\left(\tfrac{1}{6} - \tfrac{1}{3}\right) = \tfrac{\pi}{3}$$

18. $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y + \sin x)\,\mathbf{i} + (z^2 + \cos y)\,\mathbf{j} + x^3\,\mathbf{k} \;\Rightarrow$ $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2\sin t\,\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D $[x^2 + y^2 \le 1]$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.10 [ET 16.7.10] with $g(x,y) = 2xy$, $P = -2z = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$ and multiplying by -1 for the downward orientation, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA \\ &= -\iint_D (8xy^2 + 6x^3 - 1)\,dA = -\int_0^{2\pi}\int_0^1 (8r^3\cos\theta\sin^2\theta + 6r^3\cos^3\theta - 1)\,r\,dr\,d\theta \\ &= -\int_0^{2\pi}\left(\tfrac{8}{5}\cos\theta\sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right)d\theta = -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi\end{aligned}$$

20. (a) By Exercise 17.5.26 [ET 16.5.26], $\operatorname{curl}(f\nabla g) = f\operatorname{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\operatorname{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g)\cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g)\cdot d\mathbf{S}$.

(b) As in (a), $\operatorname{curl}(f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f)\cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f\nabla f)]\cdot d\mathbf{S} = 0$.

(c) As in part (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \qquad &&\text{[by Exercise 17.5.24 [ET 16.5.24]]} \\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} &&[\text{since } \mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f)\cdot d\mathbf{S} = 0$.

9. $\operatorname{div}\mathbf{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

18. As in the hint to Exercise 19, we create a closed surface $S_2 = S \cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot(-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} (-1)\,dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = \int_0^1\int_0^{2\pi}\int_1^{2-r^2} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi} (r - r^3)\,d\theta\,dr = (2\pi)\frac{1}{4} = \frac{\pi}{2}$. Thus the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

24. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (2x + 2y + z^2)\,dS$, so $\mathbf{F}\cdot\mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div}\mathbf{F} = 1$.

If $B = \{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}$, then $\iint_S (2x + 2y + z^2)\,dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

27. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ by Theorem 17.5.11 [ET 16.5.11].