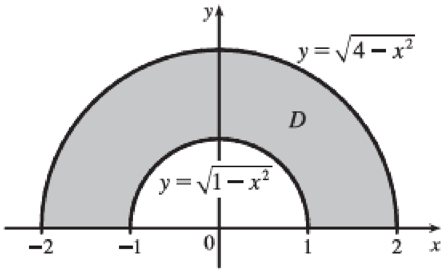
13.



$\rho(x,y) = k\sqrt{x^2+y^2} = kr,$

$$m = \iint_D \rho(x,y)dA = \int_0^\pi \int_1^2 kr \cdot r\,dr\,d\theta$$
$$= k\int_0^\pi d\theta \int_1^2 r^2\,dr = k(\pi)\left[\tfrac{1}{3}r^3\right]_1^2 = \tfrac{7}{3}\pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^\pi \int_1^2 (r\cos\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta \int_1^2 r^3\,dr$$
$$= k\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(0)\left(\tfrac{15}{4}\right) = 0$$

[this is to be expected as the region and density function are symmetric about the y-axis]

$$M_x = \iint_D y\rho(x,y)dA = \int_0^\pi \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta \int_1^2 r^3\,dr$$
$$= k\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(1+1)\left(\tfrac{15}{4}\right) = \tfrac{15}{2}k.$$

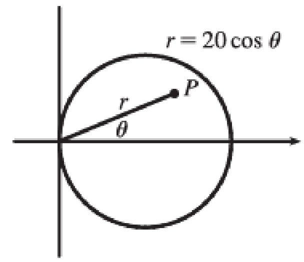
Hence $(\overline{x},\overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

33. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P, A)\,dA = k\iint_D \frac{20 - d(P, A)}{20}\,dA = k\iint_D \left[1 - \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{20}\right] dx\,dy$$

(b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k\iint_D \left[1 - \frac{1}{20}\sqrt{x^2 + y^2}\right] dx\,dy \\ &= k\int_0^{2\pi}\int_0^{10}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = 2\pi k\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_0^{10} \\ &= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k \end{aligned}$$

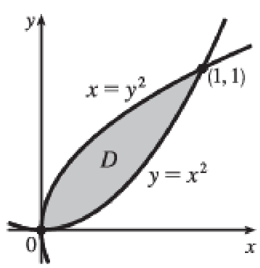


For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta} d\theta \\ &= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta \\ &= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right] \\ &= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

14.



E is the solid above the region shown in the xy-plane and below the plane $z = x + y$. Thus,

$$\begin{aligned} \iiint_E xy\,dV &= \int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}} xy(x + y)\,dy\,dx \\ &= \int_0^1\int_{x^2}^{\sqrt{x}} (x^2y + xy^2)\,dy\,dx = \int_0^1 \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{y=x^2}^{y=\sqrt{x}} dx \\ &= \int_0^1 \left(\tfrac{1}{2}x^3 + \tfrac{1}{3}x^{5/2} - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^7\right) dx \\ &= \left[\tfrac{1}{8}x^4 + \tfrac{2}{21}x^{7/2} - \tfrac{1}{14}x^7 - \tfrac{1}{24}x^8\right]_0^1 = \tfrac{1}{8} + \tfrac{2}{21} - \tfrac{1}{14} - \tfrac{1}{24} = \tfrac{3}{28} \end{aligned}$$

21. $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} dz\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5 - y - 1)\, dy\, dx = \int_{-3}^{3} \left[4y - \tfrac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$

$= \int_{-3}^{3} 8\sqrt{9-x^2}\, dx = 8\left[\tfrac{x}{2}\sqrt{9-x^2} + \tfrac{9}{2}\sin^{-1}\left(\tfrac{x}{3}\right)\right]_{-3}^{3}$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$= 8\left[\tfrac{9}{2}\sin^{-1}(1) - \tfrac{9}{2}\sin^{-1}(-1)\right] = 36\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 36\pi$

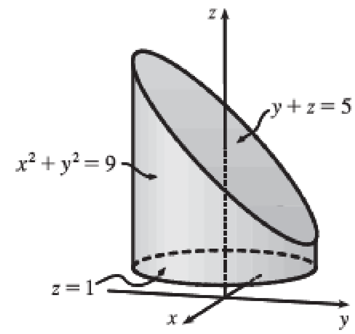
Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - y)\, dy\, dx = \int_{0}^{2\pi} \int_{0}^{3} (4 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \tfrac{1}{3}r^3 \sin\theta\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} (18 - 9\sin\theta)\, d\theta$$

$$= 18\theta + 9\cos\theta \Big]_0^{2\pi} = 36\pi$$



40. $m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[(1-x)y - y^2\right] dy\, dx$

$= \int_0^1 \left[\tfrac{1}{2}(1-x)^3 - \tfrac{1}{3}(1-x)^3\right] dx = \tfrac{1}{6}\int_0^1 (1-x)^3\, dx = \tfrac{1}{24}$

$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[(x - x^2)y - xy^2\right] dy\, dx$

$= \int_0^1 \left[\tfrac{1}{2}x(1-x)^3 - \tfrac{1}{3}x(1-x)^3\right] dx = \tfrac{1}{6}\int_0^1 \left(x - 3x^2 + 3x^3 - x^4\right) dx = \tfrac{1}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right) = \tfrac{1}{120}$

$M_{xz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[(1-x)y^2 - y^3\right] dy\, dx$

$= \int_0^1 \left[\tfrac{1}{3}(1-x)^4 - \tfrac{1}{4}(1-x)^4\right] dx = \tfrac{1}{12}\left[-\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{60}$

$M_{xy} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\tfrac{1}{2}y(1-x-y)^2\right] dy\, dx$

$= \tfrac{1}{2}\int_0^1 \int_0^{1-x} \left[(1-x)^2 y - 2(1-x)y^2 + y^3\right] dy\, dx = \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}(1-x)^4 - \tfrac{2}{3}(1-x)^4 + \tfrac{1}{4}(1-x)^4\right] dx$

$= \tfrac{1}{24}\int_0^1 (1-x)^4\, dx = -\tfrac{1}{24}\left[\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{120}$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}\right)$.

50. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}
\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\
&= C\int_0^{\infty} e^{-0.5x}\,dx \int_0^{\infty} e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\
&= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy \lim_{t\to\infty}\int_0^t e^{-0.1z}\,dz \\
&= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \\
&= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right] \\
&= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C
\end{aligned}$$

So we must have $100C = 1 \quad\Rightarrow\quad C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned}
P(X\le 1, Y\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^{\infty} \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\
&= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\
&= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\
&= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132
\end{aligned}$$

(c) $P(X\le 1, Y\le 1, Z\le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \frac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$

$$\begin{aligned}
&= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz \\
&= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1 \\
&= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787
\end{aligned}$$

8. Since $2r^2 + z^2 = 1$ and $r^2 = x^2 + y^2$, we have $2(x^2+y^2) + z^2 = 1$ or $2x^2 + 2y^2 + z^2 = 1$, an ellipsoid centered at the origin with intercepts $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm 1$.

22. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r,\theta,z) \mid 0\le\theta\le 2\pi, 0\le r\le 1, -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\right\}$. Thus the volume is

$$\begin{aligned}
\iiint_E dV &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8-3^{3/2})
\end{aligned}$$

39. The region E of integration is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. So the integral becomes

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho \sin\phi\cos\theta)\,(\rho\sin\phi\sin\theta)\,\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4} \sin^3\phi\, d\phi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{\sqrt{2}} \rho^4\, d\rho = \left(\int_0^{\pi/4} \left(1-\cos^2\phi\right)\sin\phi\, d\phi\right) \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \left(\tfrac{1}{3} - 1\right)\right] \cdot \tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}$$