4. Since $(x+y+z)^r/(x^2+y^2+z^2)$ is a rational function with domain $\{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$, f is continuous on $\mathbb{R}^3$ if and only if $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0) = 0$. Recall that $(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to $(x+y+z)^2$ gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4(x^2+y^2+z^2)$. Now for each r,

$$|(x+y+z)^r| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4(x^2+y^2+z^2)\right]^{r/2} = 2^r(x^2+y^2+z^2)^{r/2}$$

for $(x,y,z) \neq (0,0,0)$. Thus

$$|f(x,y,z) - 0| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{|(x+y+z)^r|}{x^2+y^2+z^2} \leq 2^r \frac{(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 2^r(x^2+y^2+z^2)^{(r/2)-1}$$

for $(x,y,z) \neq (0,0,0)$. Thus if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r(x^2+y^2+z^2)^{(r/2)-1} \to 0$ as $(x,y,z) \to (0,0,0)$ and so $\lim_{(x,y,z)\to(0,0,0)} (x+y+z)^r/(x^2+y^2+z^2) = 0$. Hence for $r > 2$, f is continuous on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,0,0) = x^r/x^2 = x^{r-2}$ for $x \neq 0$. So when $r = 2$, $f(x,y,z) \to 1 \neq 0$ as $(x,y,z) \to (0,0,0)$ along the x-axis and when $r < 2$ the limit of $f(x,y,z)$ as $(x,y,z) \to (0,0,0)$ along the x-axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at $(0,0,0)$ and thus is not continuous on $\mathbb{R}^3$.

8. The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is

$$y_0^2z_0^2(x - x_0) + 2x_0y_0z_0^2(y - y_0) + 2x_0y_0^2z_0(z - z_0) = 0 \quad \Rightarrow \quad (y_0^2z_0^2)x + (2x_0y_0z_0^2)y + (2x_0y_0^2z_0)z = 5x_0y_0^2z_0^2 = 5.$$

Using the formula derived in Example 13.5.8 [ET 12.5.8], we find that the distance from $(0, 0, 0)$ to this tangent plane is

$$D(x_0, y_0, z_0) = \frac{|5x_0y_0^2z_0^2|}{\sqrt{(y_0^2z_0^2)^2 + (2x_0y_0z_0^2)^2 + (2x_0y_0^2z_0)^2}}.$$

When D is a maximum, D^2 is a maximum and $\nabla D^2 = \mathbf{0}$. Dropping the subscripts, let

$f(x, y, z) = D^2 = \dfrac{25(xyz)^2}{y^2z^2 + 4x^2z^2 + 4x^2y^2}$. Now use the fact that for points on the surface $xy^2z^2 = 1$ we have $z^2 = \dfrac{1}{xy^2}$,

to get $f(x, y) = D^2 = \dfrac{25x}{\dfrac{1}{x} + \dfrac{4x}{y^2} + 4x^2y^2} = \dfrac{25x^2y^2}{y^2 + 4x^2 + 4x^3y^4}$. Now $\nabla D^2 = \mathbf{0} \quad \Rightarrow \quad f_x = 0$ and $f_y = 0$.

$$f_x = 0 \quad \Rightarrow \quad \frac{50xy^2(y^2 + 4x^2 + 4x^3y^4) - (8x + 12x^2y^4)(25x^2y^2)}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \quad \Rightarrow$$

$$xy^2(y^2 + 4x^2 + 4x^3y^4) - (4x + 6x^2y^4)x^2y^2 = 0 \quad \Rightarrow \quad xy^4 - 2x^4y^6 = 0 \quad \Rightarrow \quad xy^4(1 - 2x^3y^2) = 0 \quad \Rightarrow$$

$1 = 2y^2x^3$ (since $x = 0$, $y = 0$ both give a minimum distance of 0). Also $f_y = 0 \quad \Rightarrow$

$$\frac{50x^2y(y^2 + 4x^2 + 4x^3y^4) - (2y + 16x^3y^3)25x^2y^2}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \quad \Rightarrow \quad 4x^4y - 4x^5y^5 = 0 \quad \Rightarrow \quad x^4y(1 - xy^4) = 0 \quad \Rightarrow$$

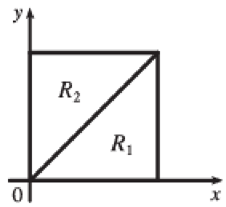
$1 = xy^4$. Now substituting $x = 1/y^4$ into $1 = 2y^2x^3$, we get $1 = 2y^{-10} \quad \Rightarrow \quad y = \pm 2^{1/10} \quad \Rightarrow \quad x = 2^{-2/5} \quad \Rightarrow$

$$z^2 = \frac{1}{xy^2} = \frac{1}{(2^{-2/5})(2^{1/5})} = 2^{1/5} \quad \Rightarrow \quad z = \pm 2^{1/10}.$$

Therefore the tangent planes that are farthest from the origin are at the four points $(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10})$. These points all give a maximum since the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$ in which case $D = 0$. The equations are

$$(2^{1/5}2^{1/5})x \pm [(2)(2^{-2/5})(2^{1/10})(2^{1/5})]y \pm [(2)(2^{-2/5})(2^{1/5})(2^{1/10})]z = 5 \quad \Rightarrow \quad (2^{2/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5.$$

2.



Let $R = \{(x, y) \mid 0 \le x, y \le 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \ge y$, and $\max\{x^2, y^2\} = y^2$ if $x \le y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$ and $R_2 = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}$. Now $\max\{x^2, y^2\} = x^2$ for $(x, y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x, y) \in R_2 \quad \Rightarrow$

$$\int_0^1\int_0^1 e^{\max\{x^2,y^2\}}\,dy\,dx = \iint_R e^{\max\{x^2,y^2\}}\,dA = \iint_{R_1} e^{\max\{x^2,y^2\}}\,dA + \iint_{R_2} e^{\max\{x^2,y^2\}}\,dA$$

$$= \int_0^1\int_0^x e^{x^2}\,dy\,dx + \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 xe^{x^2}\,dx + \int_0^1 ye^{y^2}\,dy = e^{x^2}\Big]_0^1 = e - 1$$

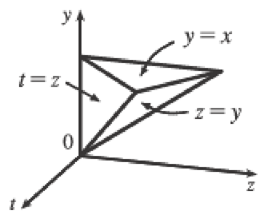
11. $\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where

$E = \{(t, z, y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$.

If we let D be the projection of E on the yt-plane then

$D = \{(y, t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram

that $E = \{(t, z, y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So



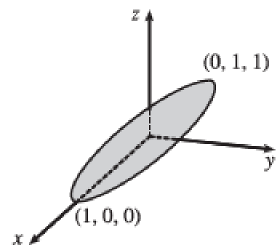
$$\begin{aligned}\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy &= \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt = \int_0^x \left[\int_t^x (y-t)\,f(t)\,dy\right] dt \\ &= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\,dt \\ &= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right) f(t)\,dt \\ &= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt\end{aligned}$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.

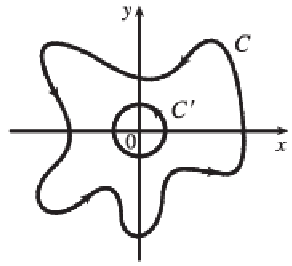


(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y} \right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot dr = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \sin^2 t - 2a \sin t}{a^2} (-a \sin t) + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$

2. By Green's Theorem

$$\int_C (y^3 - y)\, dx - 2x^3\, dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3 - y)}{\partial y} \right] dA = \iint_D (1 - 6x^2 - 3y^2)\, dA$$

Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$ can be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x, y, z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$. Now

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$= \left(\tfrac{1}{2}a + \tfrac{1}{2}a\right) \mathbf{i} + \left(\tfrac{1}{2}b + \tfrac{1}{2}b\right) \mathbf{j} + \left(\tfrac{1}{2}c + \tfrac{1}{2}c\right) \mathbf{k} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = \mathbf{n}$$

so $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S dS$ which is simply the surface area of S. Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$ is the plane area enclosed by C.