37. $4x^2 + y^2 = 16 \quad \Leftrightarrow \quad \frac{x^2}{4} + \frac{y^2}{16} = 1$. The equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \quad \Leftrightarrow$ $4x^2 + y^2 + z^2 = 16$.

44. $(1, 0, 0)$ corresponds to $t = 0$. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$, and in Exercise 4 we found that $\mathbf{r}'(t) = \langle -\sin t, \cos t, -\tan t \rangle$ and $|\mathbf{r}'(t)| = |\sec t|$. Here we can assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and then $\sec t > 0 \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sec t$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sec t} = \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle \quad \text{and} \quad \mathbf{T}(0) = \langle 0, 1, 0 \rangle.$$

$\mathbf{T}'(t) = \langle -[(\sin t)(-\sin t) + (\cos t)(\cos t)], 2(\cos t)(-\sin t), -\cos t \rangle = \langle \sin^2 t - \cos^2 t, -2\sin t \cos t, -\cos t \rangle$, so

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \frac{\langle -1, 0, -1 \rangle}{\sqrt{1 + 0 + 1}} = \frac{1}{\sqrt{2}} \langle -1, 0, -1 \rangle = \left\langle -\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}} \right\rangle.$$

Finally, $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \langle 0, 1, 0 \rangle \times \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$.

46. $t = 1$ at $(1, 1, 1)$. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$ is normal to the normal plane, so an equation for this plane is $1(x - 1) + 2(y - 1) + 3(z - 1) = 0$, or $x + 2y + 3z = 6$.

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 1, 2t, 3t^2 \rangle$. Using the product rule on each term of $\mathbf{T}(t)$ gives

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -\tfrac{1}{2}(8t + 36t^3), 2(1 + 4t^2 + 9t^4) - \tfrac{1}{2}(8t + 36t^3)2t, \\ &\qquad 6t(1 + 4t^2 + 9t^4) - \tfrac{1}{2}(8t + 36t^3)3t^2 \rangle \\ &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -4t - 18t^3, 2 - 18t^4, 6t + 12t^3 \rangle = \frac{-2}{(14)^{3/2}} \langle 11, 8, -9 \rangle \text{ when } t = 1.\end{aligned}$$

$\mathbf{N}(1) \parallel \mathbf{T}'(1) \parallel \langle 11, 8, -9 \rangle$ and $\mathbf{T}(1) \parallel \mathbf{r}'(1) = \langle 1, 2, 3 \rangle \quad \Rightarrow \quad$ a normal vector to the osculating plane is $\langle 11, 8, -9 \rangle \times \langle 1, 2, 3 \rangle = \langle 42, -42, 14 \rangle$ or equivalently $\langle 3, -3, 1 \rangle$.

An equation for the plane is $3(x - 1) - 3(y - 1) + (z - 1) = 0$ or $3x - 3y + z = 1$.

18. Let $f(x, y) = \sqrt{y + \cos^2 x}$. Then $f_x(x, y) = \frac{1}{2}(y + \cos^2 x)^{-1/2}(2\cos x)(-\sin x) = -\cos x \sin x / \sqrt{y + \cos^2 x}$ and $f_y(x, y) = \frac{1}{2}(y + \cos^2 x)^{-1/2}(1) = 1 / \left(2\sqrt{y + \cos^2 x}\right)$. Both f_x and f_y are continuous functions for $y > -\cos^2 x$, so f is differentiable at $(0, 0)$ by Theorem 8. We have $f_x(0, 0) = 0$ and $f_y(0, 0) = \frac{1}{2}$, so the linear approximation of f at $(0, 0)$ is $f(x, y) \approx f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + 0x + \frac{1}{2}y = 1 + \frac{1}{2}y$.

32. $xyz = \cos(x + y + z)$. Let $F(x, y, z) = xyz - \cos(x + y + z) = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}.$$

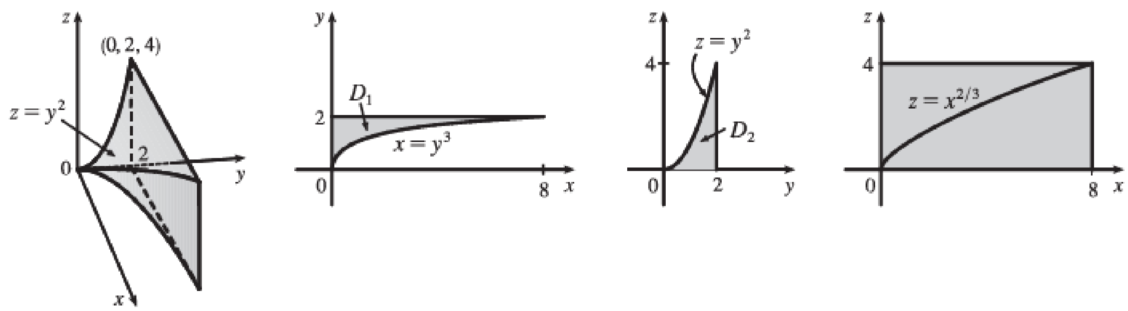
49. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is

$V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

40. The region of integration is the solid hemisphere $x^2 + y^2 + z^2 \le 4$, $x \ge 0$.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2 (\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho$$
$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi$$

46.



$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\right\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$D_1 = \left\{(x,y) \mid 0 \le y \le 2, 0 \le x \le y^3\right\} = \left\{(x,y) \mid 0 \le x \le 8, \sqrt[3]{x} \le y \le 2\right\}$,

$D_2 = \left\{(y,z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\right\} = \left\{(y,z) \mid 0 \le y \le 2, 0 \le z \le y^2\right\}$, $D_3 = \left\{(x,z) \mid 0 \le x \le 8, 0 \le z \le 4\right\}$.

Therefore we have

$$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \int_0^8\int_{\sqrt[3]{x}}^2\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{\sqrt{z}}^2\int_0^{y^3} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^2\int_0^{y^2}\int_0^{y^3} f(x,y,z)\,dx\,dz\,dy$$
$$= \int_0^8\int_0^{x^{2/3}}\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8\int_{x^{2/3}}^4\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^4\int_0^{z^{3/2}}\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4\int_{z^{3/2}}^8\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz$$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

28. $z = f(x,y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\, dS = \iint\limits_{x^2+y^2 \le 4} (x^2 + y^2)(4 + x + y)\, \sqrt{3}\, dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\, r^3 (4 + r\cos\theta + r\sin\theta)\, d\theta\, dr = \int_0^2 8\pi \sqrt{3}\, r^3\, dr = 32\pi\sqrt{3}$$

32. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\, \mathbf{i} + 2\sin t\, \mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\, \mathbf{i} + 2\cos t\, \mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\, \sin t\, \mathbf{i} + 2\sin t\, \mathbf{j} + e^{4\cos t \sin t}\, \mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t \cos t$. Thus $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t\, \sin^2 t + 4\sin t\, \cos t)\, dt = \left[-16\left(-\frac{1}{4}\sin t\, \cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$.

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2)\, r\, dz\, dr\, d\theta = 2\pi \int_0^1 (6r^3 + 8r)\, dr = 11\pi$