Problem Set 3 - Math 440/508, Fall 2011

The third homework assignment, due on Monday November 14, consists of the problems marked with asterisks.

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1. Stein and Shakarchi, page 105, Chapter 3, Section 8, Exercises 13, 14, 15, 16, 17.
2. ($*$) Stein and Shakarchi, page 108, Chapter 3, Section 9, Problem 1.
3. Stein and Shakarchi, page 109, Chapter 3, Section 9, Problem 3.
4. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
be an analytic function in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. Assume that
$$\sum_{n=2}^{\infty} n|a_n| \leq |a_1| \quad \text{ with } a_1 \neq 0.$$
Prove that f is injective.
5. ($*$) A fractional linear transformation maps the annulus $r < |z| < 1$ (where $r > 0$) onto the domain bounded by the two circles $|z - 1/4| = 1/4$ and $|z| = 1$. Find r.
6. Let f be a holomorphic function of the unit disk $\mathbb{D}$ into itself, which is not the identity map. Show that f can have at most one fixed point.
7. ($*$) Consider the polynomial
$$p(z) = z^5 + z^3 + 5z^2 + 2.$$
How many zeros (counting multiplicities) does p have in the annular region $1 < |z| < 2$?
8. (a) Let
$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$
be analytic in the unit disk. Assuming that f is injective, find the area of $f(\mathbb{D})$ in terms of the coefficients $\{c_n : n \geq 0\}$.
   (b) Compute the area of the image of the unit disk under the map $f(z) = z + \frac{z^2}{2}$.
9. Let the function f be analytic in the unit disk with $|f(z)| \leq 1$ and $f(0) = 0$. Assume that there is a number r in $(0, 1)$ such that $f(r) = f(-r) = 0$. Prove that
$$|f(z)| \leq |z| \left| \frac{z^2 - r^2}{1 - r^2 z^2} \right|.$$
10. ($*$) Suppose f is analytic in some open set containing $\overline{\mathbb{D}}$ and has the property
$$|f(z)| = 1 \quad \text{ where } \quad |z| = 1.$$
Find a formula for f in terms of its zeros in $\mathbb{D}$.

11. (a) For $|z| < 1$ define $f(z)$ by

$$f(z) = \exp\left\{-i \log\left[i\left(\frac{1+z}{1-z}\right)\right]^{\frac{1}{2}}\right\}.$$

Describe $f(\mathbb{D})$.

(b) Discuss the mapping properties of $(1-z)^i$.

12. If $\mathcal{G}$ is a group and $\mathcal{N}$ is a subgroup then $\mathcal{N}$ is said to be a normal subgroup of $\mathcal{G}$ if $S^{-1}TS \in \mathcal{N}$ whenever $T \in \mathcal{N}$ and $S \in \mathcal{G}$. A group $\mathcal{G}$ is said to be simple if the only normal subgroups of $\mathcal{G}$ are the trivial ones, namely the identity and $\mathcal{G}$ itself. Prove that the group of Möbius transformations is a simple group.

13. (a) Prove that there is no one-to-one conformal map of the punctured disc $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

(b) More generally, find a necessary and sufficient condition for two annuli, both centered at the origin, to be conformally equivalent.

14. Let Γ be a circle through points z_2, z_3, z_4. The points $z, z^* \in \mathbb{C}_\infty$ are said to be *symmetric* about Γ if

$$(z, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

(a) Give a geometric description of the relation between z and z^*. Does it agree with your notion of symmetry about a straight line?

(b) Prove the symmetry principle: If a Möbius transformation T takes a circle Γ_1 to a circle Γ_2, then any pair of points symmetric with respect to Γ are mapped by T onto a pair of points symmetric about Γ_2.

15. If Γ is a circle then an *orientation* for Γ is an ordered triple of point (z_1, z_2, z_3) such that $z_j \in \Gamma$, $j = 1, 2, 3$. (Intuitively, these points give a direction to Γ, why?). Given such an orientation, we define the right and left side of Γ with respect to (z_1, z_2, z_3) to be the sets

$$\{z \in \mathbb{C} : \operatorname{Im}(z, z_1, z_2, z_3) > 0\} \text{ and } \{z \in \mathbb{C} : \operatorname{Im}(z, z_1, z_2, z_3) < 0\}$$

respectively.

Prove the orientation principle: Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_1, z_2, z_3) be an orientation for Γ_1. Then T takes the right and left sides of Γ_1 onto the right and left sides of Γ_2 with respect to the orientation (Tz_1, Tz_2, Tz_3).