Problem Set 4 - Math 440/508, Fall 2011

The fourth homework assignment, due on Friday November 25, consists of the problems marked with asterisks.

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1. ($*$) Stein and Shakarchi, page 69, Chapter 2, problem 5.
2. Prove Vitali's theorem: Suppose G is a region and $\{f_n\} \subseteq \mathbb{H}(G)$ is locally bounded. If $f \in \mathbb{H}(G)$ has the property that
$$A = \left\{ z \in G : \lim_{n\to\infty} f_n(z) = f(z) \right\}$$
has a limit point in G, then $f_n \to f$.
3. ($*$) This problem concerns two families of functions.
   (a) Is the family of functions $\mathcal{F} = \{f_n : n \geq 1\}$ given by $f_n(z) = \tan nz$ normal on $G = \{z : \mathrm{Im}(z) > 0\}$? If not, produce a sequence that does not have a convergent subsequence. If yes, find $\overline{\mathcal{F}}$.
   (b) Let G be a region. For any $M > 0$, let $\mathcal{G}_M$ be the family of all functions in $\mathbb{H}(G)$ such that
$$\iint_G |f(x+iy)|^2 \, dx \, dy \leq M.$$
   Is $\mathcal{G}_M$ normal?
4. In class, we discussed $\mathbb{H}(G)$ as a subset of $C(G, \mathbb{C})$. We would like to carry out a similar analysis for the set of all meromorphic functions $\mathbb{M}(G)$ on G, considering it as a subset of $C(G, \mathbb{C}_\infty)$. Recall that via the stereographic projection, $\mathbb{C}_\infty$ is endowed with the chordal metric d: for $z, z' \in \mathbb{C}$,
$$d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{\frac{1}{2}}}, \qquad d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}.$$
Use d to impose a metric ρ on $C(G, \mathbb{C}_\infty)$, exactly as before; namely,
$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}, \quad \rho_n(f, g) = \sup \{ d(f(z), g(z)) : z \in K_n \},$$
where $K_n \subseteq G$ is an increasing sequence of compact sets that exhaust G, and satisfy $K_n \subseteq \mathrm{int}(K_{n+1})$.
   Verify the following statements:
   (a) Let f_n be a sequence in $\mathbb{M}(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic or $f \equiv \infty$.
   (b) $\mathbb{M}(G) \cup \{\infty\}$ is a complete metric space.
   (c) $\mathbb{H}(G) \cup \infty$ is closed in $C(G, \mathbb{C}_\infty)$.
5. ($*$) If f is a meromorphic function of the region G then define $f^\# : G \to \mathbb{R}$ by
$$f^\#(z) = \left\{ \begin{array}{ll} \dfrac{2|f'(z)|}{1 + |f(z)|^2} & \text{if } z \text{ is not a pole of } f, \\ \displaystyle\lim_{w \to z} \frac{2|f'(w)|}{1 + |f(w)|^2} & \text{if } z \text{ is a pole of } f. \end{array} \right\}$$

Prove the following theorem due to Marty:
A family $\mathcal{F} \subseteq \mathbb{M}(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if the family $\mathcal{F}^\# = \{f^\# : f \in \mathcal{F}\}$ is locally bounded.

6. $(*)$ Let $A(0; r, R)$ denote the annulus centered at the origin, with inner radius r and outer radius R, $r < R$. Find a necessary and sufficient condition for two annuli $A(0; r_1, R_1)$ and $A(0; r_2, R_2)$ to be conformally equivalent.
7. Let (X_n, d_n) be a metric space for each $n \geq 1$ and let $X = \prod_{n=1}^{\infty} X_n$. For $\xi, \eta \in X$ with $\xi = (x_n)$, $\eta = (y_n)$, $x_n, y_n \in X_n$, define

$$d^*(\xi, \eta) := \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

   (a) Show that (X, d^*) is a metric space.
   (b) Describe the convergent sequences in this metric space.
   (c) Show that if (X_n, d_n) is compact for every n, then (X, d^*) is a metric space. (Hint: Use Cantor diagonalization.)
8. Let G be a domain in $\mathbb{C}$, and (Ω, d) a complete metric space. In our discussion of normal subfamilies of $C(G, \Omega)$, we mentioned the Arzela-Ascoli theorem, which goes as follows:
Arzela-Ascoli Theorem. *A set $\mathcal{F} \subseteq C(G, \Omega)$ is normal if and only if both the following conditions hold:*
   (i) For each $z \in G$, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω;
   (ii) $\mathcal{F}$ is equicontinuous at each point in G.

   The purpose of this problem is to sketch a proof of this theorem. Fill in the details of the steps outlined below.
   (a) Assume that $\mathcal{F}$ is normal. Show that for each $z \in G$, the map from $C(G, \Omega)$ to Ω given by

$$f \mapsto f(z)$$

   is continuous. Use this to deduce part(i) of the theorem.
   (b) Show that if $\mathcal{F}$ is normal, then for every compact set $K \subset G$ and $\epsilon > 0$ there are functions $f_1, \cdots, f_n$ in $\mathcal{F}$ such that for every $f \in \mathcal{F}$ there is some $k \in \{1, \cdots, n\}$ with

$$\sup \{d(f(z), f_k(z)) : z \in K\} < \epsilon.$$

   Use this to show that $\mathcal{F}$ is equicontinuous on G.
   (c) Conversely, suppose that $\mathcal{F}$ satisfies conditions (i) and (ii). Let $\{z_n : n \geq 1\}$ be an enumeration of all points in G with rational real and imaginary parts. If $\{f_k : k \geq 1\}$ is a sequence of functions in $\mathcal{F}$, use problem 7 to deduce the existence of a subsequence $\{f_{k_j} : j \geq 1\} \subseteq \{f_k : k \geq 1\}$ such that for every $n \geq 1$,

$$\lim_{j \to \infty} f_{k_j}(z_n) = \omega_n \quad \text{for some } \omega_n \in \Omega.$$

   (d) Use the conclusion in part (c) to show that the subsequence $\{f_{k_j}\}$ is Cauchy in $(C(G, \Omega), \rho)$, and hence converges. This concludes the proof that $\mathcal{F}$ is normal.