Homework 3 - Math 440/508, Fall 2012

Due Friday November 30 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. A *finite Blaschke product* is a rational function of the form
$$B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\overline{a}_1 z}\right)\cdots\left(\frac{z-a_n}{1-\overline{a}_n z}\right)$$
where $a_1, \cdots, a_n \in \mathbb{D}$ and $0 \le \varphi \le 2\pi$. Show that any analytic function on the unit disk $\mathbb{D}$ which is continuous upto the boundary and maps $\partial\mathbb{D}$ into itself is a finite Blaschke product.

2. Describe the following transformation groups:

   (a) $\mathrm{Aut}(\mathbb{C})$ (b) $\mathrm{Aut}(\mathbb{C}_\infty)$ (c) $\mathrm{Aut}(\mathbb{D}\setminus\{0\})$ (d) $\mathrm{Aut}(\mathbb{C}\setminus\{0\})$ (e) $\mathrm{Aut}(\mathbb{H})$.

   Here as usual $\mathbb{D}$ and $\mathbb{H}$ denote the open unit disk and upper half plane respectively.

3. Determine all conformal self-maps of the complex plane with m punctures. More precisely, determine $\mathrm{Aut}(\mathbb{C}\setminus\{a_1, \cdots, a_m\})$, where $a_1, a_2, \cdots, a_m$ are m distinct points in $\mathbb{C}$. Use this to describe $\mathrm{Aut}(\Omega)$ where $\Omega = \mathbb{C}\setminus\{0,1\}, \mathbb{C}\setminus\{-1,0,1\}, \mathbb{C}\setminus\{-1,0,2\}$.

4. If $f(z)$ is analytic and satisfies $|f(z)| < 1$ for $|z| < 1$, then show that
$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}.$$
If $f(z)$ is an automorphism of $\mathbb{D}$, then verify that equality holds at every point $z \in \mathbb{D}$. Conversely, if $f \notin \mathrm{Aut}(\mathbb{D})$ then show that the inequality is strict for all $|z| < 1$.

5. Determine whether the following statements are true or false, with proper justification.

   (a) There exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$.

   (b) There exists a function f that is analytic in a region containing $\overline{\mathbb{D}}$ with the following properties: $|f(z)| = 1$ for $|z| = 1$, f has a simple zero at $z = (1+i)/4$ and a double zero at $z = 1/2$ and $f(0) = 1/2$.

(c) There is a unique analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 1/2$ and $f'(0) = 3/4$.

(d) There exists an analytic function that maps $G = \mathbb{C} \setminus [-1, 1]$ onto $\mathbb{D}$.

6. Let Γ be a circle in $\mathbb{C}_\infty$ containing z_2, z_3, z_4. Recall that z, $z^* \in \mathbb{C}_\infty$ are *symmetric* with respect to Γ if
$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

(a) Show that the definition of symmetry is independent of the choice of points z_2, z_3, z_4 in Γ.

(b) Prove the *symmetry principle* for Möbius transformations: If a Möbius transformation T takes a circle Γ_1 onto the circle Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

(c) If Γ is a circle then an *orientation* for Γ is an ordered triple of points (z_2, z_3, z_4) in Γ. If (z_2, z_3, z_4) is an orientation of Γ then we define the right and left sides of Γ (with respect to this orientation) to be
$$\{z : \operatorname{Im}(z, z_2, z_3, z_4) > 0\} \quad \text{and} \quad \{z : \operatorname{Im}(z, z_2, z_3, z_4) < 0\} \quad \text{respectively.}$$
Prove the *orientation principle*: Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_2, z_3, z_4) be an orientation for Γ_1. Then T takes the right (respectively left) side of Γ_1 onto the right (respectively left) side of Γ_2 with respect to the orientation (Tz_2, Tz_3, Tz_4).