Homework 4 - Math 541, Fall 2012

Due by 5 pm Wednesday December 5.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Complete the following outline to show that for $1 < p < \infty$, then there exists a genuine norm $|\cdot|_{p,\infty}$ on $L^{p,\infty}$ equivalent to the quasinorm $||\cdot||_{p,\infty}$. For each function f that is absolutely integrable on every set of finite measure, define
$$|f|_{p,\infty} = \sup_E \mu(E)^{-\frac{1}{p'}} \int_E |f|\, d\mu,$$
the supremum being taken over all sets satisfying $\mu(E) < \infty$.
   (a) Verify that $|\cdot|_{p,\infty}$ is a norm, and that $L^{p,\infty}(X)$ is complete under the norm.
   (b) Show that if $1 < p < \infty$, then $|f|_{p,\infty} \sim ||f||_{p,\infty}$, that is, both are finite if either is, and each is bounded by a (p-dependent) constant multiple of the other, uniformly in f.

2. Prove the following extension of Young's convolution inequality, and its consequence, fractional integration.
   (a) If $p, q, r \in (1, \infty)$ and $r^{-1} = p^{-1} + q^{-1} - 1$, then $L^p(\mathbb{R}^d) * L^{q,\infty}(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$.
   (b) The operator $f \mapsto f * |x|^{-\frac{d}{q}}$ maps $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ whenever $1 < p, q, r < \infty$ and $r^{-1} = p^{-1} + q^{-1} - 1$.

3. The Hardy-Littlewood maximal operator M is of fundamental importance in part because it controls certain other operators. The following problem illustrates this in the context of the Dirichlet problem for Laplace's equation.
   (a) Suppose that $g : \mathbb{R}^d \to [0, \infty]$ is radial and nonincreasing, i.e., $g(x) = h(|x|)$ and $h(r_1) \geq h(r_2)$ for $0 \leq r_1 \leq r_2$. Show that for any $f \geq 0$,
$$f * g(x) \leq ||g||_1 Mf(x) \quad \text{for all } x.$$
   (b) Recall the Poisson kernel
$$p_t(x) = c_d t^{-d}(1 + t^{-1}|x|)^{-\frac{d+1}{2}}.$$
   We have seen that for any $p \in [1, \infty)$, and any $f \in L^p$, $u(x, t) = f * p_t$ satisfies Laplace's equation $\Delta u = 0$, with the boundary condition $u(\cdot, t) \to f$ in L^p as $t \to 0$.

We now wish to analyze the pointwise convergence of u to f. For $r > 0$ and $x \in \mathbb{R}^d$, define the cone

$$\Gamma_r(x) = \left\{(y,t) \in \mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0,\infty) : |x-y| < rt\right\}.$$

A function v defined on $\mathbb{R}^{d+1}_+$ is said to have nontangential limit z at $x \in \mathbb{R}^d$ if for every $r > 0$, $v(y,t) \to z$ as $(y,t) \to (x,0)$ with $(y,t) \in \Gamma_r(x)$.

For any $p \in [1,\infty)$, and every $f \in L^p(\mathbb{R}^d)$, show that $u(y,t) = p_t * f(y)$ has nontangential limit $f(x)$ at almost every $x \in \mathbb{R}^d$. *Hint:* Introduce the nontangential maximal function

$$N_r u(x) = \sup_{(y,t)\in\Gamma_r(x)} \bigl|u(y,t)\bigr|.$$

4. Define the dilation operators δ_r as follows: $\delta_r f(x) = f(rx)$ for $x \in \mathbb{R}^d$ and measurable functions f. For $\phi \in \mathcal{S}'(\mathbb{R}^d)$, $\delta_r\phi \in \mathcal{S}'(\mathbb{R}^d)$ is defined by $\langle \delta_r\phi, f\rangle = r^{-d}\langle \phi, \delta_{1/r} f\rangle$. $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is said to be *homogeneous of degree* $\gamma \in \mathbb{C}$ if $\delta_r\phi \equiv r^\gamma\phi$ for all $r > 0$.

   Verify that each of the following tempered distributions is homogeneous, and determine its degree of homogeneity.

   (a) The Dirac distribution $f \mapsto f(0)$.

   (b) The principal value distribution pv k defined by

   $$\mathrm{pv}\int f(x)k(x)dx = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} f(x)k(x)\,dx, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

   where $k \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a function homogeneous of degree $-d$, satisfying

   $$\int_{\mathbb{S}^{d-1}} k(x)d\sigma(x) = 0.$$

   Here $d\sigma$ denotes the normalized surface measure on the $(d-1)$-dimensional unit sphere.

   (c) $\widehat{\phi}$ and $\partial^\alpha\phi$ if $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree γ.

5. (Homogeneous distributions continued)

   (a) If $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree γ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$, then show that $\widehat{\phi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ as well.

   (b) If $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree $-d + i\tau$ for some $\tau \in \mathbb{R}$ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$, then show that the operator $Tf = f * \phi$, initially defined from $\mathcal{S}$ to C^∞, extends to a bounded linear operator on $L^2(\mathbb{R}^d)$.

   (c) Use the preceding result to derive Lebesgue mapping properties of the operator $f \mapsto f * (\mathrm{pv}\ k)$.

(d) Let $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be homogeneous of degree zero. Then show that the operator $f \mapsto (\widehat{f}m)^\vee$ extends to a bounded linear operator on L^p for all $p \in (1, \infty)$, as well as to a bounded linear operator from L^1 to $L^{1,\infty}$.

(e) Prove the following sample application of the class of results above, often used in the theory of partial differential equations (both linear and nonlinear). For every $d \geq 2$ and $p \in (1, \infty)$ there exists $C < \infty$ such that for any $1 \leq i, j \leq d$, for all $f \in C_0^2(\mathbb{R}^d)$,

$$\left|\left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right|\right|_p \leq C||\Delta f||_p.$$