Take-home Midterm - Math 440/508, Fall 2014

Due Friday October 24 at the beginning of lecture.

Instructions: Your work will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Let f be holomorphic in an open connected set containing the annulus $\{z \in \mathbb{C} : r_1 \leq |z - z_0| \leq r_2\}$, where $0 < r_1 < r_2$.
   (a) Use an appropriate contour to obtain an integral self-reproducing formula analogous to the Cauchy integral formula for $f(z)$ in terms of the values of f on C_{r_1} and C_{r_2}. Here $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$.

   (b) Use the formula you obtained in part (a) to derive the Laurent series expansion of f:
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$
   and verify that it converges absolutely and uniformly on the annulus.

   (c) Derive integral expressions for a_n in terms of f analogous to the derivative forms of Cauchy integral formula.

2. (a) Determine whether the limit
$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix^2}\, dx$$
   exists. If yes, find its value. If not, justify why not.

   (b) By integrating a branch of $\log z/(z^3 - 1)$ around the boundary of an indented sector of aperture $\frac{2\pi}{3}$, show that
$$\int_0^\infty \frac{\log x}{x^3 - 1}\, dx = \frac{4\pi^2}{27}.$$

3. Justify the following statements.
   (a) If m and n are positive integers, then the polynomial
$$p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$$
   has exactly n zeros inside the unit disc, counting multiplicities.

(b) For any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and for $n \geq 1$, the function $(z-1)^n e^z - \lambda$ has n zeros satisfying $|z-1| < 1$ and no other zeros in the right half plane.

4. Let f be analytic in the punctured disc $G = B(a; R) \setminus \{a\}$.
(a) Show that if

$$\iint_G |f(x+iy)|^2 \, dy \, dx < \infty,$$

then f has a removable singularity at $z = a$.

(b) Suppose that $p > 0$ and

$$\iint_G |f(x+iy)|^p \, dy \, dx < \infty.$$

What can you conclude about the nature of singularity of f at $z = a$?

5. Determine whether each of the following statements is true or false. Provide a proof or a counterexample, as appropriate, in support of your answer.
(a) There exists a function f that is meromorphic on $\mathbb{C}_\infty$ such that

$$\sum_{\substack{a \in \mathbb{C}_\infty \\ \text{pole of } f}} \operatorname{Res}(f; a) \neq 0.$$

Here $\operatorname{Res}(f; a)$ denotes the residue of f at a. (Hint: By definition, $\operatorname{Res}(f; \infty) = \operatorname{Res}(\tilde{f}; 0)$, where $\tilde{f}(z) = -\frac{1}{z^2} f(\frac{1}{z})$.)

(b) The number of zeros and poles of a meromorphic function in $\mathbb{C}_\infty$ is the same, counted as always with multiplicity.

(c) For any two polynomials P and Q such that $\deg(P) \leq \deg(Q) - 2$, and Q only has simple roots, the following identity holds:

$$\sum_{a : Q(a) = 0} \frac{P(a)}{Q'(a)} = 0.$$