Midterm Solutions - Math 440/508, Fall 2014

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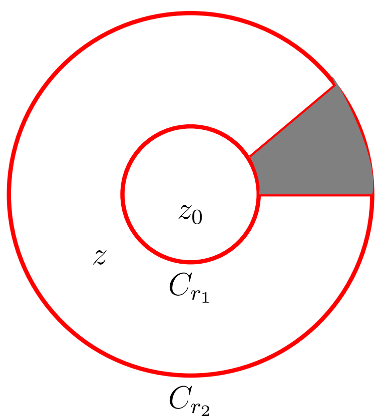
1. **Let f be holomorphic in an open connected set containing the annulus $\{z \in \mathbb{C} : r_1 \le |z - z_0| \le r_2\}$, where $0 < r_1 < r_2$.**
   (a) **Use an appropriate contour to obtain an integral self-reproducing formula analogous to the Cauchy integral formula for $f(z)$ in terms of the values of f on C_{r_1} and C_{r_2}. Here $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$.**
   (b) **Use the formula you obtained in part (a) to derive the Laurent series expansion of f:**
   $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$
   and verify that it converges absolutely and uniformly on the annulus.
   (c) **Derive integral expressions for a_n in terms of f analogous to the derivative forms of Cauchy integral formula.**



Proof. Let C_{r_1}, C_{r_2} denote the circle centered at z_0 radius r_1, r_2 respectively with $0 < r_1 < r_2$. Since C_{r_1}, C_{r_2} is compact living in the open set Ω (i.e. disjoint from the closed set Ω^C) there is some $\delta-$ neighborhood of C_{r_1}, C_{r_2} that live inside Ω. Let ρ_1, ρ_2 be such that $0 < \rho_1 < r_1 < r_2 < \rho_2$ and $C_{\rho_1}, C_{\rho_2} \subseteq \Omega$. Let S_z be the boundary of the gray sector depicted in the figure which does not contain z. Let R_z be the boundary of the remaining white region. Here all integrals over closed curve are taken **counterclockwise** which we denote by $\oint$.

Let $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$. Hence by Cauchy's theorem,

$$\oint_{S_z} g(\zeta)d\zeta = 0, \oint_{R_z} g(\zeta)d\zeta = 2\pi i f(z)$$

Here our integrals are taken in anticlockwise direction. Hence

$$f(z) = \frac{1}{2\pi i}\left[\oint_{S_Z} g(\zeta)d\zeta + \oint_{R_Z} g(\zeta)d\zeta\right] = \frac{1}{2\pi i}\left[\oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}d\zeta - \oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}d\zeta\right] \tag{1}$$

Note that $|f(\zeta)| \le M$ for some $M > 0$ on C_{ρ_2} or C_{ρ_1} which are compact. Now if $\zeta \in C_{\rho_2}$ and $|z - z_0| \le r_2$ then $|\frac{z-z_0}{\zeta-z_0}| < 1$ hence

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} = \frac{f(\zeta)}{(\zeta - z_0)(1 - \frac{z-z_0}{\zeta-z_0})} = \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

We claim that $\frac{f(\zeta)}{\zeta-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n$ converges uniformly on $\{z : |z - z_0| \le r_2\}$. To see this,if $|z - z_0| \le r_2$ we have

$$\left|\frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n\right| \le \frac{M}{r_2}\sum_{n=0}^{\infty}\left|\frac{r_2}{\rho_2}\right|^n$$

The series on RHS is convergent as $|r_2/\rho_2| < 1$ and is independent of z hence it is easy to see that the convergence is uniform in the region.

By uniform convergence (it is also uniform in $\zeta \in C_{\rho_2}$), we may swap the sum and integral:

$$\frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right)(z - z_0)^n$$

In the same fashion, suppose $|z - z_0| \ge r_1, \zeta \in C_{\rho_1}$ then

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} = -\frac{f(\zeta)}{(z - z_0)(1 - \frac{\zeta-z_0}{z-z_0})} = -\frac{f(\zeta)}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^n$$

For all such z, we have

$$\left|\frac{f(\zeta)}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^n\right| \le \frac{M}{r_1}\sum_{n=0}^{\infty}\left(\frac{\rho_1}{r_1}\right)^n$$

the series on RHS is convergent as $|\rho_1/r_1| < 1$ as is independent of z hence the series is uniformly convergent on $|z - z_0| \ge \rho_1$. Swapping the sum and the integral, one has

$$\begin{aligned}\oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}d\zeta &= -\frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^n d\zeta\right)(z - z_0)^{-(n+1)} \\ &= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\left(\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^{n-1}d\zeta\right)(z - z_0)^{-n}\end{aligned}$$

By equation (1), we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\left[\oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}d\zeta - \oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}d\zeta\right] \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{n-1}}d\zeta\right)(z-z_0)^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^{n-1}d\zeta\right)(z-z_0)^{-n} \\
&= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n
\end{aligned}$$

where

$$a_n = \begin{cases} \frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta & \text{if } n \geq 0 \\ \frac{1}{2\pi i}\oint_{C_{r_1}} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta & \text{if } n < 0. \end{cases}$$

where the series converges uniformly on $\{z : r_1 \leq |z| \leq r_2\} \subseteq \Omega$. Also we may obtain a more general formula from homotopy form of Cauchy's theorem. Two paths γ_0, γ_1 with the same endpoints inside Ω is said to be homotopic if γ_0 can be continuously deformed to γ_1 with their endpoints keep fixed. Cauchy's theorem states that if f is holomorphic in Ω then

$$\int_{\gamma_0} f(\zeta)d\zeta = \int_{\gamma_1} f(\zeta)d\zeta$$

so we can write

$$a_n = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

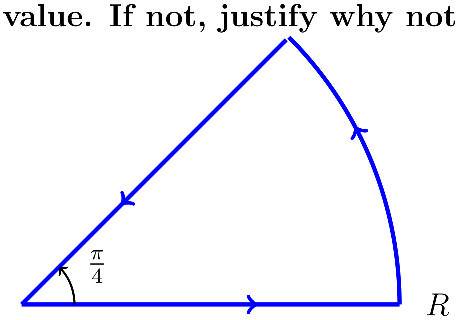
for any simple closed curve $\gamma \subseteq \Omega$, enclosing z_0.

□

2. (a) **Determine whether the limit**

$$\lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\,dx$$

exists. If yes, find its value. If not, justify why not.



Proof. Consider the positively oriented closed contour $\Gamma(R)$ consisting of the three curves

$$\Gamma_1(R) := [0, R], \qquad \Gamma_2(R) := \{Re^{i\theta} : 0 \le \theta \le \frac{\pi}{4}\},$$
$$\Gamma_3(R) := \{te^{i\frac{\pi}{4}} : 0 \le t \le R\},$$

Since the function $f(z) = e^{iz^2}$ is entire, Cauchy's theorem gives

$$\oint_{\Gamma_R} f(z)\, dz = 0,$$
$$\text{or, } I_1(R) + I_2(R) + I_3(R) = 0 \quad \text{for every } R. \tag{2}$$

Here $I_j(R)$ denotes the integral of f over $\Gamma_j(R)$ with orientation consistent with Γ. We observe that

$$\begin{aligned}
I_3(R) &= -\int_0^R e^{it^2 e^{i\frac{\pi}{2}}}\, e^{i\frac{\pi}{4}}\, dt = -\frac{1+i}{\sqrt{2}} \int_0^R e^{-t^2}\, dt \\
&\to -\frac{1+i}{\sqrt{2}} \int_0^\infty e^{-t^2}\, dt = -\frac{(1+i)}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}, \qquad \text{as } R \to \infty, \text{ whereas} \\
|I_2(R)| &= \Big| \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} iRe^{i\theta}\, d\theta \Big| \le \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} R\, d\theta \\
&\le R \int_0^{\frac{\pi}{4}} e^{-cR^2\theta}\, d\theta \le \frac{1 - e^{-dR^2}}{cR} \le \frac{2}{cR} \to 0.
\end{aligned}$$

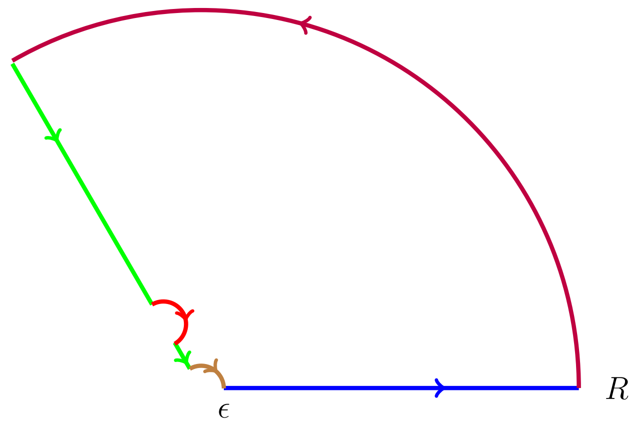
Here c and d are fixed positive constants. Inserting these estimates into (2), we obtain that

$$\lim_{R\to\infty} \int_{-R}^R e^{ix^2}\, dx = 2 \lim_{R\to\infty} I_1(R) = \frac{\sqrt{\pi}(1+i)}{\sqrt{2}}.$$

□

(b) **By integrating a branch of** $\log z/(z^3-1)$ **around the boundary of an indented sector of aperture** $\frac{2\pi}{3}$**, show that**

$$\int_0^\infty \frac{\log x}{x^3 - 1}\, dx = \frac{4\pi^2}{27}.$$



Proof. We choose the contour $\Gamma(\epsilon, R)$ to be a positively oriented closed curve, consisting of the following pieces:

$$\Gamma_1(R) = [0, R], \quad \Gamma_2(R) = \{Re^{i\theta} : 0 \le \theta \le \frac{2\pi}{3}\},$$
$$\Gamma_3(\epsilon, R) = \{te^{\frac{2\pi i}{3}} : t \in [\epsilon, 1-\epsilon] \cup [1+\epsilon, R]\},$$
$$\Gamma_4(\epsilon) = \{e^{\frac{2\pi i}{3}} + \epsilon e^{i\theta} : -\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\},$$
$$\Gamma_5(\epsilon) = \{\epsilon e^{i\theta} : 0 \le \theta \le \frac{2\pi}{3}\}.$$

Define a complex branch of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$, and note that for every $0 < \epsilon \ll 1 \ll R$, the function $f(z) = \log z/(z^3 - 1)$ is holomorphic on an open set containing $\Gamma(\epsilon, R)$ and its interior. In particular, observe that $z = 1$ is a removable singularity for f, and that the only pole of f that $\Gamma(\epsilon, R)$ approaches arbitrarily closely is $z = e^{\frac{2\pi i}{3}}$. Since $\Gamma(\epsilon, R)$ is homotopic to zero, Cauchy's theorem implies that

$$\oint_{\Gamma(\epsilon,R)} f(z)\, dz = 0. \tag{3}$$

We will show below that

$$\lim_{R\to\infty} \int_{\Gamma_2(R)} f(z)\, dz = 0, \qquad \lim_{\epsilon\to\infty} \int_{\Gamma_5(\epsilon)} f(z)\, dz = 0, \tag{4}$$

On the other hand, it is easy to verify that

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \oint_{\Gamma_3(\epsilon,R)} f(z)\, dz = -e^{\frac{2\pi i}{3}} \int_0^\infty \frac{\log x + \frac{2\pi i}{3}}{x^3 - 1}\, dx, \tag{5}$$

$$\lim_{\epsilon\to 0} \oint_{\Gamma_4(\epsilon)} = -\pi i \operatorname{Res}(f; e^{\frac{2\pi i}{3}}) = \frac{\pi^2}{9} e^{-\frac{2\pi i}{3}}(1 + i\sqrt{3}). \tag{6}$$

Assuming these for the moment, the evaluation of the integral is completed as follows. Let I denote the integral to the determined, and let J be the principal value integral given by

$$J = \lim_{\epsilon \to 0} \int_{\substack{x \in (0,\infty) \\ |x-1|>\epsilon}} \frac{dx}{x^3 - 1}\, dx.$$

. Letting $R \to \infty$ and $\epsilon \to 0$ in (3), and using (4), (5) and (6), we obtain that

$$I + 0 - e^{\frac{2\pi i}{3}}(I + \frac{2\pi i}{3} J) + \frac{\pi^2}{9} e^{-\frac{2\pi i}{3}}(1 + i\sqrt{3}) = 0.$$

Multiplying both sides of the equation above by $e^{\frac{2\pi i}{3}}$, and then equating real parts of both sides, we obtain that $\frac{3}{2} I = 2\frac{\pi^2}{9}$, or $I = \frac{4\pi^2}{27}$. □

Proof of (4). It remains to prove (4). We estimate both integrals by parametrizing the respective curves.

$$\begin{aligned}
\left| \oint_{\Gamma_2(R)} f(z)\, dz \right| &\leq \left| \int_0^{\frac{2\pi}{3}} \frac{\log(Re^{i\theta})}{R^3 e^{3i\theta} - 1} Rie^{i\theta}\, d\theta \right| \\
&\leq C \frac{R \log R}{R^3 - 1} \to 0 \text{ as } R \to \infty. \\
\left| \oint_{\Gamma_2(\epsilon)} f(z)\, dz \right| &\leq \left| \int_0^{\frac{2\pi}{3}} \frac{\log(\epsilon e^{i\theta})}{\epsilon^3 e^{3i\theta} - 1} \epsilon i e^{i\theta}\, d\theta \right| \\
&\leq C \frac{\epsilon |\log \epsilon|}{1 - \epsilon^3} \to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

□

3. **Justify the following statements.**

(a) **If m and n are positive integers, then the polynomial**

$$p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$$

has exactly n zeros inside the unit disc, counting multiplicities.

Proof. Let $p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$ then on $\{z; |z| = 1\}$ we have

$$|p(z) + (-3z^n)| = \left| 1 + z + \cdots \frac{z^m}{m!} \right| \leq \sum_{n=0}^{\infty} \frac{|z|^m}{m!} \leq e^{|z|} < 3$$

Hence for $|z| = 1, |p(z) + (-3z^n)| < 3 \leq |p(z)| + |-3z^n|$. By Rouche's Theorem, $p(z)$ and $-3z^n$ have the same number of zeros in $\{z : |z| < 1\}$ and it is easy to see that $-3z^n$ has n roots (counted with multiplicities) with $|z| < 1$, the same is true for $p(z)$. *Note: You can't compare two complex numbers, just their absolute values.* □

(b) **For any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and for $n \geq 1$, the function $(z-1)^n e^z - \lambda$ has n zeros satisfying $|z-1| < 1$ and no other zeros in the right half plane.**

Proof. Let $f(z) = (z-1)^n e^z - \lambda, |\lambda| < 1$ and $g(z) = -(z-1)^n e^z$. Let $\Omega = \{z : |z-1| < 1\}$ On $\partial\Omega$ we have $\Re(z) > 0$ and $f(z) \neq 0$. Hence

$$|f(z) + g(z)| = |\lambda| < 1 \leq | - (z-1)^n e^z = |g(z)| < |f(z)| + |g(z)|$$

By Rouche's Theorem, f, g have the same number of zeros inside Ω. Since e^z has no zeros , it is easy to see that g has n zeros (counted with multiplicities) inside Ω and hence the same is true for f.

On $\{\Re(z) > 0\} \cap \{|z-1| > 1\}$, we have $|(z-1)^n e^z| = |z-1|^n e^{\Re(z)} \geq 1e^0 > |\lambda|$ hence no other zeros on the right half plane. ☐

4. **Let f be analytic in the punctured disc $G = B(a; R) \setminus \{a\}$.**

(a) **Show that if**

$$\iint_G |f(x+iy)|^2 \, dy \, dx < \infty,$$

then f has a removable singularity at $z = a$.

(b) **Suppose that $p > 0$ and**

$$\iint_G |f(x+iy)|^p \, dy \, dx < \infty.$$

What can you conclude about the nature of singularity of f at $z = a$?

Proof. WLOG, assume $a = 0$. Suppose f has a pole of order n at 0 then $f(z) = \frac{g(z)}{z^n}$ for some g holomorphic on $B(0,R), g(0) \neq 0$. For $\epsilon > 0$ small enough we have $C_2 > |g(z)| > C_1 > 0$ on $B(0,\epsilon)$ for some C_1, C_2. Now using polar coordinate,

$$\begin{aligned}
\int\int_{B(0,\epsilon)\setminus\{0\}} |\frac{g(x,y)}{z^n}|^p dxdy < \infty &\iff \int\int_{B(0,\epsilon)\setminus\{0\}} \frac{1}{|z|^{np}} dxdy < \infty \\
&\iff \int_0^{2\pi} \int_0^{\epsilon} r^{-np} r dr d\theta < \infty \\
&\iff -np + 1 > -1 \iff p < \frac{2}{n}
\end{aligned}$$

Hence f has a pole of order n which is the biggest positive integer that is less than $2/p$.

If f has a essential singularity at 0 then by considering its Laurent's Series, we may find a sequence of functions g_N which has a pole of order N at 0 and $g_N \to f$ uniformly on $\delta \leq |z| \leq \epsilon$ for any $\delta > 0$. Hence as $N \to \infty$

$$\int_{\delta \leq |z| \leq \epsilon} |g_N|^p \to \int_{\delta \leq |z| \leq \epsilon} |f|^p$$

But when N is large enough so that $p \geq \frac{2}{N}$ then

$$\limsup_{M\to\infty} \int_{\frac{1}{M}\leq|z|\leq\epsilon} |g_N|^p = \infty$$

This would imply

$$\limsup_{M\to\infty} \int_{\frac{1}{M}\leq|z|\leq\epsilon} |f|^p = \infty$$

□

so f cannot have essential singularity at 0. In particular if $p = 2$, we must have a removable singularity there. We conclude that the integral of $|f|^p$ on G cannot converge for any $p > 0$ if f has an essential singularity at a.
Note: We may not have uniform convergence $B(0, R)\backslash\{0\}$ so we may not directly interchange the limit and the integral directly on this domain.

5. **Determine whether each of the following statements is true or false. Provide a proof or a counterexample, as appropriate, in support of your answer.**
   (a) **There exists a function f that is meromorphic on $\mathbb{C}_\infty$ such that**

$$\sum_{\substack{a\in\mathbb{C}_\infty \\ \textbf{pole of } f}} \mathbf{Res}(f; a) \neq 0.$$

Here $\mathbf{Res}(f; a)$ denotes the residue of f at a. (Hint: By definition, $\mathbf{Res}(f;\infty) = \mathbf{Res}(\tilde{f}; 0)$, where $\tilde{f}(z) = -\frac{1}{z^2}f(\frac{1}{z})$.)

Proof. The statement is false. By the residue theorem and a change of variable $w = 1/z$, we see that for any meromorphic function f on $\mathbb{C}_\infty$ and a constant $R > 0$ sufficiently large,

$$\mathrm{Res}(f, \infty) = -\oint_{|z|=R^{-1}} \frac{1}{z^2} f\Big(\frac{1}{z}\Big)\, dz = -\oint_{|w|=R} f(w)\, dw = -\sum_{\substack{a\in\mathbb{C} \\ a \text{ pole of } f}} \mathrm{Res}(f, a).$$

Hence the sum of the residues of a meromorphic function on the extended complex plane is always zero. □

   (b) **The number of zeros of a meromorphic function in $\mathbb{C}_\infty$ is the same as the number of poles, both counted with multiplicity.**

Proof. The statement is true. A meromorphic function in $\mathbb{C}_\infty$ is a rational function P/Q, where P and Q are polynomials with no common root. Let m and n denote the degrees of P and Q respectively. Then P/Q has exactly m zeros and n poles in $\mathbb{C}$, counting multiplicities. If $m = n$, these account for all zeros and poles of P/Q in $\mathbb{C}_\infty$, and the statement has been verified. If $m < n$, then P/Q has a zero of multiplicity

$n - m$ at ∞, whereas if $m > n$, then P/Q has a pole of order $m - n$ at ∞. Thus the total number of zeros in $\mathbb{C}_\infty$ matches the total number of poles in all cases. $\square$

(c) **For any two polynomials P and Q such that deg$(P) \leq$ deg$(Q) - 2$, and Q only has simple roots, the following identity holds:**

$$\sum_{\mathbf{a}:\mathbf{Q}(\mathbf{a})=\mathbf{0}} \frac{\mathbf{P}(\mathbf{a})}{\mathbf{Q}'(\mathbf{a})} = \mathbf{0}.$$

Proof. The statement is true. Since Q has only finitely many zeros, choose $R > 0$ large enough so that $B(0; R)$contains all the roots of Q. Further these roots are simple, hence the rational function P/Q has at most simple poles, located at the roots of Q. By the residue theorem,

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{P(z)}{Q(z)}\, dz = \sum_{a:Q(a)=0} \operatorname{Res}(a; P/Q). \tag{7}$$

On one hand, if a is a root of Q, then

$$\operatorname{Res}(a; P/Q) = \lim_{z\to a} (z-a)\frac{P(z)}{Q(z)} = \lim_{z\to a} \frac{P(z)}{\frac{Q(z)-Q(a)}{z-a}} = \lim_{z\to a} \frac{P(a)}{Q'(a)}. \tag{8}$$

On the other hand, the assumption $\deg(P) \leq \deg(Q) - 2$ implies that

$$\int_{|z|=R} \frac{P(z)}{Q(z)}\, dz \leq C R^{\deg(P)-\deg(Q)+1} \to 0 \text{ as } R \to \infty. \tag{9}$$

Combining (7), (8) and (9) yields the desired claim. $\square$