Chapter 1, Exercise 5

A set Ω is said to be *pathwise connected* if any two points in Ω can be joined by a (piecewise-smooth) curve contained entirely in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

Part (a)

Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint, non-empty open sets. Chhose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a Parameterization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t^* = \sup_{0 \le t \le 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \le s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$

Solution

As suggested, we consider the point $z(t^*)$. We ask the question: which of Ω_1 and Ω_2 contains this point? Evidently, this point is not in Ω_1: if $z(t^*)$ is in Ω_1, then, because Ω_1 is open, there is an open ball B containing $z(t^*)$. Since z is continuous, it follows that $z^{-1}(B)$ is open as a subset of $[0,1]$. Thus (assuming $t^* < 1$) $z^{-1}(\Omega_1)$ contains points to the right of t^*, which is impossible. If $t^* = 1$, then there is a sequence of points in Ω_1 that converges to $z(1) \in \Omega_2$, contradicting the assumption that Ω_2 is open.

If we assume instead that $z(t^*) \in \Omega_2$, we recognize that $z(t) \in \Omega_2$ if and only if $t > t^*$. Thus t^* is the infimum of all values of t such that $z(t) \in \Omega_2$, and we can use the same argument as in the previous paragraph to conclude $z(t^*) \notin \Omega_2$. Since $z(t^*) \in \Omega_1 \cup \Omega_2$, this is a contradiction.

0.1 Part b

Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1, Ω_2 are open, disjoin, and their union is Ω. Finally, since Ω_1 is nonempty (why?) conclude that $\Omega = \Omega_1$ as desired.

0.1.1 Solution

Evidently $\Omega_1 \cup \Omega_2 = \Omega$ and Ω_1 is disjoint from Ω_2. The only thing that remains to be shown is that both Ω_1 and Ω_2 are open.

Let $w_1 \in \Omega_1$. Because Ω is open, Ω contains an open ball B centered at w_1. It is obvious that if $w^* \in B$, then there is a path z^* connecting w_1 and w^*. Let z_1 be a curve joining w to w_1. Then consider the curve defined by

$$z(t) = \begin{cases} z(2t) & \text{if } 0 \leq t < 1/2 \\ z(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then z is a continuous, piecewise smooth curve that connects w to w^*. It follows that $B \subset \Omega_1$ and that Ω_1 is open.

Now, let $w_2 \in \Omega_2$. Because Ω is open Ω contains an open ball B centered at w_2. Let $w^* \in B$. If there were a curve z_2 that connected w to w^*, then we could, as in the previous paragraph, find a curve connecting w to w_2 by concatenating the path from w to w^* and the path from w^* to w_2. Thus $w_2 \in \Omega_1$, which is a contradiction.

Thus Ω can be written as $\Omega_1 \cup \Omega_2$ for disjoint open sets Ω_1 and Ω_2. Since Ω is connected, either $\Omega_1 = \Omega$ or $\Omega_2 = \Omega$. But $w \in \Omega_1$, so Ω_1 is nonempty and therefore $\Omega_1 = \Omega$.

Chapter 1, Exercise 7

The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called Blaschke factors, will reappear in the various applications in later chapters.

Part a

Let z, w be two complex numbers such that $\bar{z}\omega \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can we assume that z is real? It then suffices to prove that

$$(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})$$

with equality for appropriate r and $|w|$.]

Solution

Write $z = re^{i\theta}$. Then

$$\begin{aligned}\left|\frac{w-z}{1-\bar{w}z}\right| &= \left|\frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}}\right| \\ &= \left|e^{i\theta}\frac{we^{-i\theta}-r}{1-\overline{we^{-i\theta}}r}\right| \\ &= \left|\frac{we^{-i\theta}-r}{1-\overline{we^{-i\theta}}r}\right|.\end{aligned}$$

Letting $w^* = we^{-i\theta}$ this becomes

$$\left|\frac{w^*-r}{1-w^*r}\right|$$

so it is enough to consider the case in which $z = r$ is a real number. Note further that replacing w by $\overline{w}$ is equivalent to taking the complex conjugate of the entire fraction. So it is enough to show

$$\left(\frac{w-r}{1-wr}\right)\left(\frac{\bar{w}-r}{1-\bar{w}r}\right) \leq 1$$

or equivalently that

$$(w-r)(\bar{w}-r) \leq (1-wr)(1-\bar{w}r).$$

Suppose first that w and r both have absolute value less than 1. Let $w = se^{i\theta}$. Pull out $e^{i\theta}$ and $e^{-i\theta}$ from the first and second factor on the left turns the left side into $(s-r)^2$. Doing the same on the right side turns the expression to $(1-sr)^2$. Since $s, r < 1$, we have that $sr < \min(|s|, |r|) \leq \max(|s|, |r|) < 1$ so $(s-r)^2$ is clearly smaller than $(1-sr)^2$ and we are done.

If s is instead equal to 1, then $s - r = 1 - r = 1 - sr$, and if $r = 1$, then $s - r = s - 1 = -(1-s) = -(1-sr)$, so we have equality in these cases.

Part b

Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-wz}$$

satisfies the following conditions:

1. F maps the unit disc to itself (that is $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic
2. F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.
3. $|F(z)| = 1$ if $|z| = 1$.
4. $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

Solution

(i) and (iii) directly follow from part (a) of the problem except for the holomorphy, which is clear except when $z\bar{w} = 1$. This can only happen if $|w| = 1$ and $z = \frac{1}{w}$. It is seen that F has a removable singularity at $z = \frac{1}{w}$ with value w. (ii) follows by plugging in: the numerator is clearly 0 when $z = w$, and plugging in $z = 0$ makes the numerator equal to w and the denominator equal to 1. All that remains to be seen is that F is bijective on $\mathbb{D}$. Consider $F \circ F(z)$. This is

$$\frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}}.$$

We simplify this:

$$\begin{aligned}
&\frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} \\
&= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \frac{\bar{w}(w-z)}{1-\bar{w}z}} \\
&= \frac{w(1-\bar{w}z) - (w-z)}{q - \bar{w}z - \bar{w}(w-z)} \\
&= \frac{w - |w|^2 z - w + z}{1 - \bar{w}z - |w|^2 + \bar{w}z} \\
&= \frac{z(1-|w|^2)}{1-|w|^2} \\
&= z
\end{aligned}$$

so the function F is an involution and therefore bijective on $\mathbb{D}$.

Chapter 1, Exercise 13

Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

1. $\mathrm{Re}(f)$ is constant;
2. $\mathrm{Im}(f)$ is constant;
3. $|f|$ is constant; one can conclude that f is constant.

Solution

Suppose that $\mathrm{Re}(f)$ is constant. Then $f(x, y) = a + iv(x, y)$ for $z = x + iy$. Then we consider the PDEs from the Cauchy-Riemann equations:

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

So $\frac{\partial v}{\partial y}$ is zero, and thus v depends only on x and

$$0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so $\frac{\partial v}{\partial x}$ is zero and thus v depends only on y. Since v cannot depend on either x or y, it follows that v is constant.

The same argument works if $\mathrm{Im}(f)$ is constant. Alternatively, if $\mathrm{Im}(f)$ is constant, then $\mathrm{Re}(if)$ is constant and so if, and thus f, is constant.

Now suppose $|f|$ is constant. Writing $f(z) = u(x,y) = iv(x,y)$ for $z = x+iy$, we then have that $u(x,y)^2 + v(x,y)^2$ is constant. In particular, this implies that $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}$ and that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$ Thus we can again use the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$

and

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

So we get $\frac{\partial u}{\partial x}$ is equal to both $\frac{\partial v}{\partial x}$ and $-\frac{\partial v}{\partial x}$, showing that both are equal to zero, and by the same logic as before, f is constant.