Chapter 2, Exercise 11

Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

- Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\mathrm{Re}\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right)\,d\phi$$

- Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}$$

Solution

Part a

We first notice that, because $R > |z|$, we have that $\frac{f(\zeta)}{\zeta - R^2/\bar{z}}$ is a holomorphic function on the disc D_R, so

$$\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - R^2/\bar{z}}\,d\zeta = 0.$$

Therefore, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\frac{1}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta - z} + \frac{1}{\zeta\bar{\zeta}/\bar{z} - \zeta}\right) \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta - z} + \frac{\bar{z}}{\zeta(\bar{\zeta} - \bar{z})}\right) \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\frac{\zeta\bar{\zeta} - z\bar{z}}{\zeta(\zeta - z)(\bar{\zeta} - \bar{z})}
\end{aligned}$$

But on the other hand:

$$\begin{aligned}
\mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) &= \frac{1}{2}\left(\frac{\zeta + z}{\zeta - z} + \frac{\bar{\zeta} + \bar{z}}{\bar{\zeta} - \bar{z}}\right) \\
&= \frac{1}{2}\frac{\zeta\bar{\zeta} - \zeta\bar{z} + z\bar{\zeta} - z\bar{z} + \zeta\bar{\zeta} - z\bar{\zeta} + \zeta\bar{z} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} \\
&= \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})}
\end{aligned}$$

So the two integrals are equal (remembering the Jacobian factor $d\phi = \frac{d\zeta}{\zeta}$).

Part b

We have from before

$$\text{Re}\left(\frac{\zeta+z}{\zeta-z}\right)=\frac{\zeta\bar{\zeta}-z\bar{z}}{(\zeta-z)(\bar{\zeta}-\bar{z})}$$

let $\zeta=Re^{i\gamma}$ and $z=r$:

$$\begin{aligned}\text{Re}\left(\frac{Re^{i\gamma}+z}{Re^{i\gamma-z}}\right)&=\frac{R^2-r^2}{(Re^{i\gamma}-r)(Re^{-i\gamma}-r)}\\&=\frac{R^2-r^2}{R^2-2Rr\cos(\gamma)+r^2},\end{aligned}$$

as desired.

Chapter 2, Exercise 12

Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is a twice continuously differentiable function and harmonic, that is,

$$\delta u(x,y)=0.$$

for all $(x,y)\in\mathbb{D}$.

- Prove that there exists a holomorphic function f on the unit disc such that

  $$\text{Re}(f)=u.$$

  Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.

- Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: if u is harmonic in the unit disc and continuous on its closure, then if $z=re^{i\theta}$, one has

  $$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-\phi)u(e^{i\phi})\,d\phi$$

  Where P_r is the Poisson kernel.

Solution

Part a

Consider the function $f(x,y)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$. Notice that this function satisfies $\frac{\partial \text{Re}f}{\partial x}=\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial \text{Im}f}{\partial y}=-\frac{\partial^2 u}{\partial y^2}$, which is equal to $\frac{\partial^2 u}{\partial x^2}$ since u is harmonic. Furthermore, $\frac{\partial \text{Re}f}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ and $-\frac{\partial \text{Im}f}{\partial y}=\frac{\partial^2 u}{\partial x\partial y}$. These are equal because f is continuously differentiable and therefore u satisfies equality of mixed partial derivatives. Therefore, f satisfies the Cauchy-Riemann equations and has an antiderivative F.

This antiderivative is necessarily of the form $u + iv(x, y)$ for some x, y since $\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x}$. So we have the equation

$$\frac{\partial \mathrm{Re} f}{\partial x} + i\frac{\partial \mathrm{Im} f}{\partial x} = u_x - iu_y.$$

So equating real parts, we get $u_x = \frac{\partial \mathrm{Re}(F)}{\partial x}$. But the real part of this derivative of F is $\frac{\partial u}{\partial x}$. Similarly, $\frac{\partial F}{\partial z} = -i\frac{\partial F}{\partial y}$, but $-i\frac{\partial F}{\partial y} = -i\frac{\partial \mathrm{Re}(F)}{\partial y} + \frac{\partial \mathrm{Im} F}{\partial y}$. Thus, equating the imaginary parts, we get $\frac{\partial \mathrm{Re}(F)}{\partial y} = u_y$. Thus the real part of F is u plus some real constant. This can be chosen to be 0 by selecting a suitable antiderivative for F. Similarly, we can look at the imaginary part of the first equation and the real part of the second equation to get a pair of partial differential equations that clearly determine $\mathrm{Im}(F)$ up to an additive real constant.

Part b

We would like to apply the result from exercise 11, but we cannot take $R = R_0 = 1$ there. Nonetheless, we have for all $R < 1$ that

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\phi})\mathrm{Re}\left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\right) d\zeta$$

Writing $z = re^{i\theta}$ and dividing the numerator and denominator by $e^{i\theta}$:

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\phi})\mathrm{Re}\left(\frac{Re^{i(\phi-\theta)} + r}{Re^{i(\phi-\theta)} - r}\right)$$

Thus

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\phi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}.$$

Taking real parts,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\phi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}.$$

for all $R < 1$.

Now, we observe what happens as $R \to 1$. Evidently the limit of the left side of the equation is $u(re^{i\theta})$ since none of the quantities on the left hand side of the equation depend on R. The denominator of the right hand size is larger than the strictly positive number $(R - r)^2$, so the dominated convergence theorem applies since the right hand side is uniformly bounded. Thus we can pull the limit inside the integral and replace R by 1.

Chapter 2, Exercise 14

Suppose f is holomorphic in a neighbourhood Ω of the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

represents the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Solution

Since $f(z)$ has a pole at z_0, we have

$$f(z) = \frac{b_{-n}}{(z-z_0)^n} + \cdots + \frac{b_{-1}}{z-z_0} + g(z),$$

where $g(z)$ is holomorphic on Ω and b_{-n} is not zero.

Thus, letting c_m be the coefficient of z^m in the expansion of $\frac{b_{-n}}{(z-z_0)^n} + \cdots + \frac{b_{-1}}{z-z_0}$ and letting d_m be the coefficient of z_m in the expansion of $g(z)$, we get

$$a_m = c_m + d_m$$

so

$$\frac{a_m}{a_{m+1}} = \frac{c_m + d_m}{c_{m+1} + d_{m+1}}$$

note that $d_m \to 0$ as $m \to \infty$, so as long as the c_m do not go to zero, we have

$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \frac{c_{m+1}c_m}{.}$$

We will consider the expansion of

$$\frac{1}{(z-z_0)^j}$$

around 0. Notice that this is a constant multiple of the derivative of

$$\frac{1}{(z-z_0)^{j-1}}.$$

Since the coefficient of z_m in the expansion of

$$\frac{1}{z-z_0} = \frac{1}{-z_0}\frac{1}{1-\frac{z}{z_0}}$$

is equal to $-(z_0)^{-m+1}$ by the geometric series formula. It follows that the limit of the ratios of the coefficients in this expansion is z_0, and that the terms in this expansion do not approach zero.

Furthermore, taking a derivative simply multiplies the coefficient of z^m by m and shifts it to the coefficient of z^{m-1}, taking any number of derivatives cannot spoil the limit of the ratios of the coefficients. Thus, for any linear combination of any derivatives of $\frac{1}{z-z_0}$ we have that the limit of the ratios of the coefficients is z_0. Furthermore, because b_{-n} is nonzero, the coefficients in the expansion of $\frac{b_{-n}}{(z-z_0)^n} + \cdots + \frac{b_{-1}}{z-z_0}$ cannot possibly approach 0 as $m \to \infty$ because the coefficients in the expansion of $\frac{1}{(z-z_0)^n}$ grow like m^j. Therefore, the limit of the c_m is nonzero and the limit of $\frac{c_m}{c_{m+1}}$ is z_0 as $m \to \infty$, as desired.