Chapter 8, Exercise 13

The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

- Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

  $$\rho(f(z), f(w)) \leq \rho(z, w) \quad \text{for all } z, w \in \mathbb{D}.$$

  Moreover, prove that if f is an automorphism of $\mathbb{D}$, then f preserves the pseudo-hyperbolic distance

  $$\rho(f(z), f(w)) = \rho(z, w) \quad \text{for all } z, w \in \mathbb{D}.$$

- Prove that

  $$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D}.$$

  This result is called the Schwarz-Pick lemma.

Solution

Part a

Suppose that f is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. Define the automorphism

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

This is a Blaschke factor as described in chapter 1, exercise 7, and as such we showed it was an automorphism of $\mathbb{D}$ in a previous assignment.

Note that ϕ_w maps w to 0, and $\phi^{-1}(w)$ maps 0 to w. Thus

$$\phi_{f(w)} \circ f \circ \phi_w^{-1}$$

maps 0 to 0. Thus we can apply the Schwarz-Pick lemma to conclude that

$$|\phi_{f(w)} \circ f \circ \phi_w^{-1}(z)| \leq |z| \quad \text{for all } z \in \mathbb{D}.$$

Fix $z \in \mathbb{D}$. Because ϕ_w^{-1} is an automorphism, there is a unique $z' = \phi_w(z)$ such that $\phi_w^{-1}(z') = z$; namely $z' = \phi_w(z)$. Thus

$$|\phi_{f(w)} \circ f(z)| \leq |z'| = \left| \frac{z - w}{1 - \bar{w}z} \right|$$

as desired.

If f is an automorphism of $\mathbb{D}$, then the same argument as above applies to f^{-1}. Thus we get, for all $\tilde{z}, \tilde{w} \in \mathbb{D}$:

$$\rho(f^{-1}(\tilde{z}), f^{-1}(\tilde{w})) \leq \rho(\tilde{z}, \tilde{w})$$

and letting $\tilde{z}$, $\tilde{w}$ be $f(z), f(w)$ we get

$$\rho(z, w) \leq \rho(f(z), f(w))$$

as desired.

Part b

In order to prove the Schwarz-Pick lemma, we use a limiting argument.

By the inequality from Part a, we have that for any holomorphic function f, and any z, w in the disc $\mathbb{D}$.

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}z} \right| \le \left| \frac{z - w}{1 - \bar{w}z} \right|$$

We will rearrange this inequality:

$$\left| \frac{f(z) - f(w)}{z - w} \frac{1}{1 - \overline{f(w)}z} \right| \le \frac{1}{1 - \bar{w}z}$$

Taking the limit as $z \to 0$ and using the differentiability of f, we get:

$$\frac{|f'(z)|}{|1 - |f(z)|^2|} \le \frac{1}{|1 - |z|^2|}$$

and the absolute values surrounding the denominators on both sides can be removed because $|z|^2 < 1$ by assumption.

Chapter 8, Problem 3

The Schwarz-Pick lemma (see Exercise 13) is the infinitesimal version of an important observation in complex analysis and geometry.

For complex numbers $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the **hyperbolic length** of w at z by

$$\|w\|_z = \frac{|w|}{1 - |z|^2},$$

where $|w|$ and $|z|$ denote the usual absolute values. This length is sometimes referred to as the **Poincaré metric**, and as a Riemann metric it is written as

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

The idea is to think of w as a vector lying in the tangent space at z. Observe that for a fixed w, its hyperbolic length grows to infinity as z approaches the boundary of the disc. We pass from the infinitesimal hyperbolic length of tangent vectors to the global hyperbolic distance between two points by integration.

- Given two complex numbers z_1 and z_2 in the disc, we define the **hyperbolic distance** between them by

$$d(z_1, z_2) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt,$$

where the infimum is taken over all smooth curves $\gamma : [0, 1] \to \mathbb{D}$ joining z_1 and z_2. Use the Schwarz-Pick lemma to prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2) \quad \text{for any } z_1, z_2 \in \mathbb{D}.$$

In other words, holomorphic functions are distance-decreasing in the hyperbolic metric.

- Prove that automorphisms of the unit disc preserve the hyperbolic distance; namely

  $$d(\phi(z_1), \phi(z_2)) = d(z_1, z_2) \quad \text{for any } z_1, z_2 \in \mathbb{D}.$$

  Conversely, if $\phi : \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance, then either ϕ or $\bar{\phi}$ is an automorphism of $\mathbb{D}$.

- Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism ϕ such that $\phi(z_1) = 0$ and $\phi(z_2) = s$ for some s on the segment $[0, 1)$ on the real line.

- Prove that the hyperbolic distance between 0 and $s \in [0, 1)$ is

  $$d(0, s) = \frac{1}{2} \log \frac{1+s}{1-s}.$$

- Find a formula for the hyperbolic distance between any two points in the unit disc.

Solution

Part a

Let γ be a curve connecting z_1 and z_2. Then we will consider:

$$\begin{aligned}
&\int_0^1 \|(f \circ \gamma)'(t)\|_{f(\gamma(t))} \, dt \\
&= \int_0^1 \frac{|(f \circ \gamma)'(t)|}{1 - |f(\gamma(t))|^2} \, dt \\
&= \int_0^1 \frac{|f'(\gamma(t))\gamma'(t)|}{1 - f(\gamma(t))^2} \, dt \\
&\leq \int_0^1 \frac{\gamma'(t)}{1 - \gamma(t)^2} \, dt.
\end{aligned}$$

Thus the integral along $f \circ \gamma$ is less than or equal to the integral along γ for any curve γ connecting z_1 and z_2. Thus, the infimum over such curves $f \circ \gamma$ is less than or equal to the infimum of the integral over such curves γ. Because the curves $f \circ \gamma$ are a subset of those curves connecting $f(z_1)$ and $f(z_2)$, the infimum over all such curves is less than or equal to the infimum over curves of the form $f \circ \gamma$.

Part b

The Schwarz-pick lemma is seen to hold with equality when $f : \mathbb{D} \to \mathbb{D}$ is an automorphism by the same argument as in part b of exercise 13, and applying the argument form part a proves the equality.

Now to tackle the converse. By part d, to be shown later, or by the rotational symmetry of the hyperbolic distance function about the origin, the level sets of the distance to the origin are circles centered at the origin, so it is immediately clear that such any such function ϕ that fixes the origin must preserve circles centered at the origin.

Now, we know that automorphisms preserve d, so it follows that the level sets of d must be circles because automorphisms of $\mathbb{D}$ are Möbius transformations. Thus f preserves circles. Furthermore, it is clear that f is continuous as can be seen by observing that f must map very small circles to very small circles.

By part c, we can locate automorphisms ψ, ϕ such that $g := \psi \circ f \circ \phi$ satisfies $g(0) = 0$ and $g(1/2) = s$ for some real s. Recalling that circles around 0 are preserved, and that circles around $\frac{1}{2}$ map to specific circles around s, we can locate $g(z)$ (up to complex conjugation) for all z satisfying $0 < \mathrm{Re}(z) < \frac{1}{2}$ by picking appropriate circles centered at 0 and $\frac{1}{2}$, and observing that the intersection points much map to the intersection points. Using this and continuity, we get that g or $\bar{g}$ (and thus f or $\bar{f}$) must be holomorphic for $0 < \mathrm{Re}(z) < \frac{1}{2}$. The choice of $1/2$ can e changed to any number between -1 and 1 in order to conclude that f must be holomorphic.

From here, it's easy to see that f is an automorphism: g is a holomorphic function that preserves circles centered at the origin, so it follows from the Schwarz lemma that f is an automorphism.

Part c

Such an automorphism can be constructed by mapping z_1 to 0 using the Blaschke factor ψ_{z_1}. Then, by choosing an appropriate θ, the negative of the argument of $\psi_{z_1}(z_2)$, we have that $e^{i\theta}\psi_{z_1}(z_2)$ is equal to some nonnegative real number s. Of course, that real number is the absolute value

$$\left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|$$

of $\psi_{z_1}(z_2)$.

Part d

We need to estimate

$$\int_0^1 \frac{|\gamma'(t)|}{1 + |\gamma(t)|^2} \, dt$$

But $\gamma'(t)$ is $x(t) + iy(t)$, and $|\gamma(t)|^2$ is $x(t)^2 + y(t)^2$. Thus the integrand is bounded by $\frac{|x'(t)}{1+|x(t)|^2}$, and we can bound our integral by

$$\int_0^1 \frac{|x'(t)}{1 - x(t)^2}\, dt$$

This integral is clearly minimized if $x'(t)$ does not change sign and no back-tracking occurs:

$$\int_0^1 \frac{x'(t)}{1 - x(t)^2}\, dt$$

We can now make a change of variables:

$$\int_0^s \frac{dx}{1 - x^2}$$

which can be computed using partial fraction decomposition.

The result is

$$d(0, s) = \frac{1}{2} \log \frac{1+s}{1-s}.$$

Part e

The hyperbolic distance is invariant under automorphisms, so we combine c and d to move z_1 to 0 and z_2 to $\left|\frac{z_1 - z_2}{1 - z_1 \bar{z}_2}\right|$. Then we get the expression

$$d(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left|\frac{z_1 - z_2}{1 - z_1 \bar{z}_2}\right|}{1 - \left|\frac{z_1 - z_2}{1 - z_1 \bar{z}_2}\right|}.$$