Math 440/508 Quiz 12 Solution

Name: **SID #:**

1. Show that there exists an analytic function f defined on the unit disc $\mathbb{D}$ centred at the origin such that $f(\mathbb{D}) = \mathbb{D} \setminus \{0\}$, and f' never vanishes on $\mathbb{D}$. Explain why this does not contradict the Riemann mapping theorem.

(10 points)

Solution. There exists a Möbius transformation T that maps $\partial\mathbb{D}$ onto $\mathbb{R}$, with $T(\mathbb{D}) = \mathbb{H}$, the upper half-space. The mapping $\varphi(z) = e^{-z}$ then maps $\mathbb{H}$ onto $\mathbb{D} \setminus \{0\}$. The function $f = \varphi \circ T$ is the desired analytic map, since $f'(z) = T'(\varphi(z))\varphi'(z) = -e^{-z}T'(\varphi(z)) \neq 0$ on $\mathbb{D}$. Recall that T is an automorphism of $\mathbb{C}_\infty$, hence has nonvanishing derivative everywhere.

We know that $\mathbb{D}$ cannot be conformally equivalent to $\mathbb{D} \setminus \{0\}$. However, the mapping f is not conformal, since φ is $2\pi i$-periodic, hence many-to-one. Thus there is no contradiction with the Riemann mapping theorem. $\square$