Math 440/508 Quiz 2 Solution, Fall 2017

1. For all possible values of a_0 and a_1, find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n \qquad \text{where} \qquad a_n = a_{n-1} + a_{n-2} \text{ for all } n > 1.$$

(10 points)

Solution. **The possible values of the radius of convergence are $(\sqrt{5} \pm 1)/2$ and ∞.**

Set $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Clearly the radius of convergence $R = \infty$ if $a_0 = a_1 = 0$. If at least one of a_0 and a_1 is nonzero, then the recursion relation above implies

$$\sum_{n=2}^{\infty} a_n z^n = \sum_{n=2}^{\infty} a_{n-1} z^n + \sum_{n=2}^{\infty} a_{n-2} z^n \quad \textit{i.e.}, f(z) - a_0 - a_1 z = z(f(z) - a_0) + z^2 f(z).$$

Solving the equation leads to the following expression for f:

$$f(z) = \frac{-a_0 + (a_0 - a_1)z}{z^2 + z - 1}. \tag{1}$$

In other words f is a rational function, which is holomorphic at all points z except possibly where $z^2 + z - 1 = 0$, i.e., $z = (-1 \pm \sqrt{5})/2$. Set

$$\alpha_1 = \frac{-1 + \sqrt{5}}{2}, \qquad \alpha_2 = \frac{-1 - \sqrt{5}}{2}, \qquad \text{so that } |\alpha_1| < |\alpha_2|.$$

The radius of convergence of f is determined by the possible cancellation of the numerator and denominator in (1). More precisely, two cases arise.

Case 1: either $a_0 = a_1$ or $a_0/(a_0 - a_1) \neq \alpha_1$. In this case f fails to be holomorphic at α_1, and hence the power series expansion is valid on the smallest disc centred at the origin excluding α_1. In other words, $R = |\alpha_1|$.

Case 2: $a_0/(a_0 - a_1) = \alpha_1$. In this case, $f(z) = (a_0 - a_1)/(z - \alpha_2)$, so its power series expansion is valid on $|z| < R = |\alpha_2|$. □