Math 440/508 Quiz 7 Solution

Name: **SID #:**

1. Suppose that f is meromorphic on an open set containing $\overline{\mathbb{D}}$, the closure of the unit disk. Assume that f does not vanish on $\partial\mathbb{D}$, and that

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} g(z)\frac{f'(z)}{f(z)}\,dz = 0$$

for all functions g that are holomorphic on $\overline{\mathbb{D}}$. What can you say about the zeros and poles of f in $\mathbb{D}$?

(10 points)

Solution. We will prove that f has no zeros or poles in $\mathbb{D}$.

Aiming for a contradiction, let us assume that $\mathcal{Z} = \{z_1, \cdots, z_M\}$ and $\mathcal{P} = \{p_1, \cdots, p_N\}$ are respectively the distinct zeros and poles of f in $\mathbb{D}$. Set a_j (resply b_k) to be the order of z_j (resply p_k). Then there exists an analytic function F not vanishing anywhere on $\mathbb{D}$ such that

$$f(z) = \prod_{j=1}^{M}(z - z_j)^{a_j}\prod_{k=1}^{M}(z - p_k)^{-b_k}F(z).$$

As we saw in the proof of the argument principle, this means

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{M}\frac{a_j}{z - z_j} - \sum_{k=1}^{N}\frac{b_k}{z - p_k} + \frac{F'(z)}{F(z)}.$$

Multiplying both sides of the equation above by a holomorphic function g and integrating over $\partial\mathbb{D}$, we find that

$$\begin{aligned} 0 = \oint_{\partial\mathbb{D}} g(z)\frac{f'(z)}{f(z)}\,dz &= \sum_{j=1}^{M}\oint_{\partial\mathbb{D}} g(z)\frac{a_j}{z - z_j} - \sum_{k=1}^{N}\oint_{\partial\mathbb{D}} g(z)\frac{b_k}{z - p_k} + \oint_{\partial\mathbb{D}}\frac{g(z)F'(z)}{F(z)} \\ &= 2\pi i\left[\sum_{j=1}^{M} a_j g(z_j) - \sum_{k=1}^{N} b_k g(p_k)\right], \end{aligned}$$

where the last step follows from Cauchy's theorem and the Cauchy integral formula.

Now fix an index j, and choose g to be a polynomial that vanishes at every point in $\mathcal{Z}$ and $\mathcal{P}$ except z_j. Then the above computation shows that $2\pi i a_j g(z_j) = 0$, which is a contradiction since a_j is by definition a positive integer and $g(z_j) \neq 0$ by our choice of g. This shows that $\mathcal{Z} = \emptyset$. The proof for $\mathcal{P}$ is identical. □