Math 440/508 Quiz 9 Solution

Name: **SID #:**

1. Show that a conformal map preserves angles, in the following sense. If $f : U \to V$ is conformal, and Γ_1, Γ_2 are two curves in U intersecting at z_0, then the angle between Γ_1 and Γ_2 at z_0 is the same as the angle between $f \circ \Gamma_1$ and $f \circ \Gamma_2$ at $f(z_0)$. (*Hint: Recall that the angle between two curves at an intersection point is, by definition, the angle between their tangents.*)

(10 points)

Solution. Let $\gamma_i : [0,1] \to U$ be a parametrization of the curve Γ_i with $\gamma_i(t_i) = z_0$. The angle θ between Γ_1 and Γ_2 at z_0 is then the angle between the vectors $\gamma_i'(t_i)$, hence

$$\theta = \arg(\gamma_1'(t_1)) - \arg(\gamma_2'(t_2)).$$

The corresponding angle between the image curves $f \circ \gamma_i$ at $f(z_0)$ is, by the same argument

$$\begin{aligned}\theta' &= \arg((f \circ \gamma_1)'(t_1)) - \arg((f \circ \gamma_2)'(t_2)) \\ &= \arg(f'(z_0)\gamma_1'(t_1)) - \arg(f'(z_0)\gamma_2'(t_2)) \\ &= \arg(f'(z_0)) + \arg(\gamma_1'(t_1)) - [\arg(f'(z_0)) + \arg(\gamma_2'(t_2))] = \theta.\end{aligned}$$

Note that the second step follows from the chain rule, while the penultimate step uses the fact that $f'(z_0) \neq 0$ (since f is conformal), as a result of which $\arg(f'(z_0))$ is well-defined (even if multi-valued). □