**Math 320 Assignment 1**
**Due Wednesday, September 12 at start of class**

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Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Please staple your pages together when you submit your assignment.

(iv) Do not forget to include your name and SID.

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1. Prove that there is no rational number whose square is 23.

2. (a) If $x, y \in \mathbb{R}$, $x < y$, then Theorem 1.20(b) of the textbook shows that there exists $p \in \mathbb{Q}$ such that $x < p < y$. Show that there is also an irrational $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$.

   (b) Given $x < y$, show that there are, in fact, infinitely many distinct rationals between $x$ and $y$.

   (c) Prove that the same is true for irrationals too.

3. We say that an ordered set $X$ obeys the *least upper bound axiom* if any nonempty subset of $X$ with an upper bound also admits a least upper bound that lies in $X$. Show that $\mathbb{Z}$ obeys the least upper bound axiom but $\mathbb{Q}$ does not.

4. Read the section on Fields, pp.5–8.
   In this problem we study a set that satisfies the field axioms but does not satisfy the axioms of an ordered field. Consider the field $\mathbb{F}_3$. This field has three elements, which we will call $0, 1, 2$. (Do not confuse these elements with real numbers: $0, 1$ are the elements prescribed to exist by axioms (A4) and (M4), and 2 is an arbitrary name for a third element.) Addition and multiplication are defined by the following addition and multiplication tables:

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| × | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

   Using a proof by contradiction, show that it is impossible to define an operation "$<$" that makes $\mathbb{F}_3$ into an ordered field with the specified operations. Hint: Proposition 1.18(d).

   *Remark.* $\mathbb{F}_3$ is an example of a finite field. Finite fields play an important role in algebra, number theory, and computer science.

5. In Theorem 1.19, the real numbers $\mathbb{R}$ are constructed as an ordered field with the least-upper-bound property. Find the sup and inf of each of the following sets of real numbers:

   (a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where $p, q, r$ each take on all positive integer values.

   (b) $E = \{x : 3x^2 - 10x + 3 < 0\}$.

(c) $E = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$, where $a < b < c < d$.

6. Let $S_1$ and $S_2$ be nonempty subsets of $\mathbb{R}$ that are bounded above. Let $S_1+S_2 = \{x+y : x \in S_1, y \in S_2\}$ and $S_1 - S_2 = \{x-y : x \in S_1, y \in S_2\}$. For each of the following statements, give a proof if it is true or a counterexample if it is false.

   (a) $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$.

   (b) If $S_2$ is also bounded below then $\sup(S_1 - S_2) = \sup S_1 - \sup S_2$.