Math 320 Assignment 2
Due Wednesday, September 19 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

---

1. Let S be the set of all finite length strings of letters from the alphabet a–z. Let $<$ be the lexicographic order on S. This means that if $w = a_1a_2...a_k$ and $v = b_1b_2...b_\ell$ are two strings, then $w < v$ if either of the following two things hold (A): $k < \ell$ and $a_i = b_i$ for each $i = 1, \ldots, k$, or (B) The letter a_j comes before b_j in the alphabet, where j is the smallest index where $a_i \neq b_i$. Thus for example, a $<$ aa, aa $<$ b, and b $<$ cde.

   (a) Let $E \subset S$ be the set of all finite strings that begin with the character a. What is the least upper bound for E? Prove that your answer is correct.

   (b) Does S have the least upper bound property? If so, prove it. If not, find an example showing that S does not have the least upper bound property and prove that your example is correct.

2. (a) Let F be a field that has finitely many elements (i.e. when considered as a set, F has finitely many elements). Prove that it is impossible to define an operation "$<$" that makes F into an ordered field.

   (b) Let $F = \{(a, b) : a, b \in \mathbb{R}\}$ be the set of ordered pairs of real numbers. We define the operations $+$ and $\cdot$ on F by

$$(a, b) + (a', b') = (a + a',\ b + b'),$$
$$(a, b) \cdot (a', b') = (aa' - bb',\ ab' + ba').$$

   With these operations, F is a field whose "0" is $(0, 0)$ and whose "1" is $(1, 0)$ (you do not need to prove this). The astute reader might observe that this field is usually referred to as $\mathbb{C}$, the field of complex numbers. Prove that it is impossible to define an operation "$<$" that makes F into an ordered field. Hint: if "$<$" is an order on F, try comparing $(0, 1)$ and $(0, 0)$.

3. Let $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

   (a) Prove that F is a field under the usual operations of addition $+$ and multiplication $\cdot$ (i.e. prove that all of the field axioms are satisfied). This field is sometimes called $\mathbb{Q}(\sqrt{2})$.

   (b) Prove that if "$<$" is the usual ordering, then F becomes an ordered field.

   (c) Prove that with this choice of ordering, F does not have the LUB property.

4. Prove that $f(x) = x$ is the only function $f : \mathbb{Q} \to \mathbb{R}$ satisfying the following properties:

   - f is an injection.
   - For all $a, b \in \mathbb{Q}$, $f(a + b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$.

   As a hint to get you started, think about $f(1)$.