Math 320 Assignment 5
Due Wednesday, October 10 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

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1. The extended complex plane $\mathbb{C}_\infty$ consists of the complex plane $\mathbb{C}$ together with an additional point ∞. A representation of the extended complex plane as a metric space can be obtained via the *stereographic projection* π, as follows. Consider the unit sphere

$$\mathbb{S}^2 = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\}.$$

Denote by $\mathbb{E}$ the infinite equatorial plane in $\mathbb{R}^3$ given by $\xi_3 = 0$, so that $\mathbb{E}$ can be identified with $\mathbb{C}$ via the correspondence: $(\xi_1, \xi_2, 0) \in \mathbb{E} \longleftrightarrow \xi_1 + i\xi_2 \in \mathbb{C}$. Write $p \in \mathbb{E}$ as $p = \pi(\xi)$, where ξ is the intersection of $\mathbb{S}^2$ with the line passing through p and the north pole $N = (0, 0, 1)$. We define ∞ in $\mathbb{C}_\infty$ by setting $\infty = \pi(N)$. Draw the picture.

(a) Show that the point $(\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$ on the sphere corresponds to the point $z = (\xi_1 + i\xi_2)/(1 - \xi_3) \in \mathbb{C}$.

(b) Conversely, given $z \in \mathbb{C}$, show that the corresponding point on $\mathbb{S}^2$ is (ξ_1, ξ_2, ξ_3) with

$$\xi_1 + i\xi_2 = \frac{2z}{1 + |z|^2}, \qquad \xi_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

(c) We define a metric on $\mathbb{C}_\infty$ by setting the distance between two complex numbers equal to the chordal distance between their representatives on $\mathbb{S}^2$. Show that this gives:

$$d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \qquad d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

(This does define a metric, but you need not verify this.)

2. Let E be a subset of a metric space X. Prove that the following statements are equivalent:

(a) E is dense in X. (See Definition 2.18(j).)

(b) The only closed set which contains E is X.

(c) The only open set disjoint from E is the empty set.

(d) E intersects every non-empty open set.

(e) E intersects every neighbourhood in X.

3. Let $a, b \in \mathbb{R}$, $a < b$. Let $C[a,b]$ denote the collection of all continuous, real-valued functions defined on the closed interval $[a,b]$.

   (a) Check that
   $$d(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)|$$
   defines a metric on $C[a,b]$.

   (b) Is the set $\mathcal{P}$ of all polynomials in the metric space $(C[a,b], d)$ an open set in $C[a,b]$?

   (c) Is $\mathcal{P}$ a closed subset of the metric space $C[a,b]$?