Math 320 Assignment 6
Due Wednesday, October 24 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

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1. Let (X, d) be a metric space. Suppose that every subset of X is compact. Prove that X must be finite.

2. Let $\mathbb{S}^2 = \{\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$; we will think of this set as a subset of $\mathbb{R}^3$, where $\mathbb{R}^3$ has the usual Euclidean metric.

   Let $(\mathbb{C}_\infty, d)$ be the extended complex plane from Homework 5, problem 1, with the metric defined in that problem.

   Let $\pi \colon \mathbb{S}^2 \to \mathbb{C}_\infty$ be the stereographic projection, and let π^{-1} be its inverse (you proved in HW 5 that π is a bijection, so in particular π^{-1} exists).

   (a) Prove that a set $G \subset \mathbb{S}^2$ is relatively open in $\mathbb{S}^2$ if and only if $\pi(G)$ is open in $\mathbb{C}_\infty$.

   (b) Prove that $(\mathbb{C}_\infty, d)$ is compact.

3. Consider the metric space $(C[a, b], d)$ from Homework 5 problem 3. Is this metric space compact? Prove that your answer is correct.

4. Let (X, d) be a metric space and let $a \in X,\ B \subset X$. Define $d(a, B) = \inf\{d(a, b) \colon b \in B\}$.

   (a) Consider $\mathbb{R}^k$ with the Euclidean metric. Let $B \subset \mathbb{R}^k$ be nonempty and compact, and let $a \in B^c$. Prove that there exists $b \in B$ such that $d(a, b) = d(a, B)$.

   (b) Consider $\mathbb{R}^k$ with the Euclidean metric. Let $B \subset \mathbb{R}^k$ be nonempty and closed, and let $a \in B^c$. Prove that there exists $b \in B$ such that $d(a, b) = d(a, B)$.

   (c) Consider $\mathbb{Q}$ with the usual Euclidean metric $d(p, q) = |p - q|$. Give an example of a nonempty closed subset $B \subset \mathbb{Q}$ and a rational number $a \in B^c$ such that there is no $b \in B$ for which $d(a, b) = d(a, B)$.