Math 320, Fall 2018
Midterm 2 Solutions

Name: SID:

Instructions

- The total time is 50 minutes.
- The maximum score is 100 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Partial credit will be assigned to the clarity and presentation style of solutions. Please ensure that your answers are effectively comunicated.
- No clarification will be given for any problems; if you believe a problem is ambiguous, interpret it as best you can and write down any assumptions you feel are necessary.

Problem	Points	Score
1	35	
2	25	
3	25	
4	15	
MAX	100	

1. a) **State the *Heine-Borel Theorem*, and give an example.**

(10 points)

Solution. Let (X, d) denote the metric space $\mathbb{R}^n$ equipped with the standard Euclidean metric. The Heine-Borel theorem states that a set is compact in (X, d) if and only if it is closed and bounded.

For example, the unit interval $[0, 1]$ is compact in $\mathbb{R}$. □

b) **State the definition for what it means for a set E in a metric space X to be *connected*, and give an example.**

(10 points)

Solution. Let (X, d) be a metric space. We say that a set $E \subseteq X$ admits a nontrivial separation if there exist sets nonempty sets A and B such that

$$E = A \cup B, \quad \overline{A} \cap B = \emptyset, \quad A \cap \overline{B} = \emptyset.$$

A set E is said to be connected in it does not admit any nontrivial separation.

The unit interval $[0, 1]$ is connected in $\mathbb{R}$. □

c) **Let $\{p_n\}$ be a sequence of real numbers. State the definition of**

$$\limsup_{n\to\infty} p_n.$$

Give an example of a sequence that is not eventually constant, and compute $\limsup$ **for that sequence.**

(15 points)

Solution.

$$\limsup_{n\to\infty} p_n = \sup\Big\{\alpha : \alpha \text{ is a limit point of } \{p_n\}\Big\}.$$

Example 1: The lim sup of any convergent sequence is its limit. Thus

$$\limsup \frac{1}{n} = 0.$$

Example 2: Another example of the lim sup of a non-convergent sequence is

$$\limsup_{n\to\infty} p_n = \limsup_{n\to\infty} (-1)^n\left(1 + \frac{1}{n}\right) = 1.$$

$\square$

2. **Let X be the set of all infinite binary strings (i.e. the set of all infinite sequences whose entries are either 0 or 1). Given elements $b = (b_1, b_2, \ldots)$ and $b' = (b'_1, b'_2, \ldots)$ of X, define**

$$d(b, b') = \sup\Big\{\sum_{k=1}^{n} 2^{-k}|b_k - b'_k| : n \in \mathbb{N}\Big\} = \sum_{k=1}^{\infty} 2^{-k}|b_k - b'_k|.$$

(X, d) is a metric space (you do not need to prove this). Is this metric space complete? Prove that your answer is correct.

(25 points)

Solution. We argue that X is complete, i.e., every Cauchy sequence in X converges.

Let $\{b^{(k)} : k \geq 1\}$ denote a Cauchy sequence in X, i.e.,

$$d\big(b^{(k)}, b^{(\ell)}\big) = \sum_{n=1}^{\infty} 2^{-n}|b_n^{(k)} - b_n^{(\ell)}| \to 0 \text{ as } k, \ell \to \infty.$$

We need to determine the limit of this sequence.

For each $n \geq 1$,

$$2^{-n}|b_n^{(k)} - b_n^{(\ell)}| \leq d\big(b^{(k)}, b^{(\ell)}\big) \to 0 \text{ as } k, \ell \to \infty.$$

In other words, for each $n \geq 1$, the sequence $\{b_n^{(k)} : k \geq 1\}$ is a Cauchy sequence consisting only of two elements 0 or 1. Hence it must be eventually constant, hence

$$b_n := \lim_{k\to\infty} b_n^{(k)} \quad \text{exists.}$$

We now proceed to show that $b \in X$ given by $b := (b_1, b_2, \cdots, b_n, \cdots)$ is the limit of the Cauchy sequence $\{b^{(k)} : k \geq 1\}$. Fix $\epsilon > 0$.

Choose $N, K \geq 1$ so that

$$\sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} < \frac{\epsilon}{2}, \text{ and} \tag{1}$$

$$|b_n^{(k)} - b_n| < \epsilon/2 \text{ for all } n \leq N \text{ and } k \geq K. \tag{2}$$

Combining (1) and (2) with the fact that $|b_n^{(k)} - b_n| \leq 1$ leads to the following estimate: for all $k \geq K$,

$$\begin{aligned} d(b^{(k)}, b) &= \sum_{n=1}^{\infty} 2^{-n} \big| b_n^{(k)} - b_n \big| \\ &\leq \sum_{n=1}^{N} 2^{-n} |b_n^{(k)} - b_n| + \sum_{n=N+1}^{\infty} 2^{-n} \\ &\leq \sum_{n=1}^{N} 2^{-n} \frac{\epsilon}{2} + 2^{-N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $b = \lim_{k\to\infty} b^{(k)}$, as claimed. □

3. **Prove that every uncountable subset of $\mathbb{R}$ has a limit point.**

Hint: If $S \subset \mathbb{R}$ is uncountable, it might be helpful to consider $S \cap [-n, n]$.

(25 points)

Proof. We know that a countable union of countable sets is countable. Since S can be written as the countable union

$$S = \bigcup_{n=1}^{\infty} S \cap [-n, n],$$

at least one of the sets $S \cap [-n, n]$ must be uncountable. This is an infinite subset of the compact set $[-n, n]$, and thus has a limit point. □

4. **Let (X, d) be a metric space and let $\{p_n\}$ be a sequence in X. Consider the set of subsequences of $\{p_n\}$, i.e. the set**

$$\big\{\{q_n\} \text{ a sequence in } X,\ \{q_n\} \text{ is a subsequence of } \{p_n\}\big\}.$$

Prove that this set cannot be countably infinite, i.e. it must either be finite or uncountable.

Hint: it might be helpful to consider the following two cases: either $\{p_n\}$ is eventually constant, or it isn't.

(15 points)

Solution. Case 1: $\{p_n\}$ is eventually constant. Let us say that $p_n = p$ for all $n \geq N$. Thus the distinct elements in the sequence can occur only in the first N slots, and are therefore at most N in number. A subsequence of $\{p_n\}$ is obtained by choosing an ordered subset (which could be empty) out of these first N elements, and adding a constant string of p. Thus the possible number of distinct subsequences is at most 2^N, which is finite.

Case 2: $\{p_n\}$ has infinitely many distinct elements. Without loss of generality (after passing to a subsequence if necessary), we may assume that no element in $\{p_n\}$ is repeated. Let $\mathcal{A}$ denote the collection of all infinite binary strings that contain infinitely many 1-s. We know that $\mathcal{A}$ is uncountable. Further, each $a = (a_1, a_2, \cdots) \in \mathcal{A}$ generates a subsequence $\{q_n\}$ of $\{p_n\}$ as follows:

$$q_n = p_{a_n}.$$

This gives rise to uncountably many distinct subsequences.

Case 3: $\{p_n\}$ has finitely many distinct elements, but is not eventually constant. In this case, one can find two numbers α and β that occur in the sequence $\{p_n\}$ infinitely often. Then all possible binary strings consisting of α and β are subsequences of $\{p_n\}$. The collection of such strings is of uncountable cardinality, completing the proof. $\square$